Homework 6

Initial response due by noon on Thursday, October 22

In this assignment, we ask that you choose from one of the following topics and write a 2-3 page paper about it. The paper should present the material in a readable manner, keeping in mind our discussion Monday of *motiviation*, as well as guiding text and connectivity. The *order of presentation* of material, use of *examples*, and use of *informal and formal language* should be your main focus for this paper. The intended audience is a technical one, e.g., a fellow mathematician who happens to pick up your paper.

You paper will be peer reviewed, meaning a fellow classmate will read and comment on your writing. Because of this, we are trying to get an even spread of topics. To this end we are asking that you submit ***by Thursday*** your list of preferences amongst the problems (i.e. rank them 1-4, 1 being most preferred, 4 being least). Topics will be given out on a first come first serve basis. Your first drafts are due ***by Monday***.

- **The completion of a metric space:** Every metric space X has a *completion*. The idea behind the definition is that the completion of a metric space should be a metric space which is complete, but not too much bigger than X. Take as your definition that the completion of a metric space X is a metric space X^* such that X^* is complete, $X \subseteq X^*$, and X is dense in X^*. Prove the existence of a completion for any metric space. You can find helpful hints/guides in Exercises 3.23 and 3.24. However, be aware that the order of the exercises is not ideal for presenting the topic.
- **The Baire category theorem:** There is a very famous and important theorem due to Baire.

  Theorem. *Let X be a complete metric space, and $\{G_\alpha\}$ a countable collection of dense open subsets of X. Then $\bigcap_\alpha G_\alpha$ is dense in X.*

  You can see Rudin's Exercises 3.21 and 3.22 for some assistance. You may wish to consider some attempts at a counterexample to aid your proof. A very mild modification of Exercise 3.22 will give a proof.
- **Sphere Neighborhoods:** For any point $x \in \mathbb{R}^n$ let C_x be the sphere (*not* ball) of radius 1 centered at x. For any subset

$X \subset \mathbb{R}^n$, let C_X be the union all of the spheres centered at points of X, i.e.

$$C_X = \bigcup_{x \in X} C_x$$

Show that if X is compact then C_X is also compact. It may be helpful to consider one or both alternate definitions of compactness suggested by Rudin's Theorem 2.41 (Heine-Borel) rather than the open cover definition we originally used.

- **Rearrangements:** A convergent series that is *not* absolutely convergent is said to be *conditionally convergent.* If $\{a_n\}_{n\in\mathbb{N}}$ is a sequence, a *rearrangement* is a sequence $\{b_n\}_{n\in\mathbb{N}}$ consisting of the same numbers as $\{a_n\}$ but in a different order. Prove the following surprising theorem of Dirichlet:

  Theorem. *Let* $\sum a_n$ *be conditionally convergent, and let* L *be any real number. Then there is a rearrangement* $\{b_n\}$ *of* $\{a_n\}$ *such that* $\sum b_n = L$.

  Rudin has a discussion of this theorem on pages 76 to 77. However, his theorem is somewhat more general, and (we find that) his proof is difficult to follow. Your proof should be more motivated.

Because this assignment requires substantial effort, you do not have to submit a solution to your regular 100B/C problem set to us this week.