Writing an 18.821 project report (Paul Seidel, Fall 2009)

What a report is *not*:

- It is *not* an exposition of known results from books and papers;
- It is *not* a homework writeup;
- It is *not* a research paper.

Instead, the aim is to *explain the problem*, *motivate your approach*, and *present your findings and insights*, for an audience which is mathematically informed (at the level of this class) but has not heard of the subject before. Think of it as a technical report, or internal working paper.

Whatever style of writing achieves these three goals is the right one. However, there are guidelines suggested by experience, and following those will make your task easier. There are also a few specific requirements for the class.

At some easily found point in the report, you need to state who's taking responsibility for each part.

Coloring knots

A. Moose (Sections 1,2,5), A. Rabett (Section 1), A. Chicken (Sections 2–4).

An abstract (optional) outlines the contents of the paper in a few sentences. It does not need to be a complete explanation. For instance:

Abstract. We study knots in three-space through their plane projections. We introduce rules for coloring the arcs in such a projection, and prove that this gives rise to a knot invariant, which can distinguish the trefoil from the unknot.

The introduction *gets the reader involved in the subject*, *summarizes your main findings in informal language*, and *prepares the reader for the following material*.

- Does it give the reader a good intuitive grasp of what the problem is?
- After reading it, will the reader know what to expect in the paper, and how the paper is organized?

1. Introduction (First version, suffers from blabla)

In this project, we investigate knots. The main question in knot theory is: *how can we tell whether a knot can be deformed into another one?* Many knot theorists have devoted a lot of time to classifying knots. Their work gave rise to the Rolfsen table of Prime knots [1]. Many invariants (the *Alexander polynomial*, the *Jones polynomial*, the *knot group*, and *knot colorings*) have been developed for this purpose.

The structure of our paper is as follows. In the next section, we explain knots and knot projections, as well as the Reidemeister moves. Section 3 introduces the notion of knot 3-coloring, and proves that 3-colorability is unchanged under Reidemeister moves. Section 4 generalizes this to n-colorability for any n. Section 5 discusses example computations. Section 6 explains how making certain modifications to a knot affects the number of 3-colorings.

Questions to ask about the introductory section:

- Does it give the reader a good intuitive grasp of what the problem is?
- Does it express your approach to the problem?

1. Introduction (Second version, somewhat abrupt)

This paper studies knots and knot colorability. Our main result is this:

Theorem. (Theorem 4.3 in the paper) *3-colorability of a knot is invariant under Reidemeister moves.*

This means that it is an invariant of the knot. For instance, one can use this to prove that the trefoil cannot be continuously deformed into the unknot: it is genuinely knotted. In fact, we will prove the stronger result that *the number of 3-colorings is a knot invariant.*

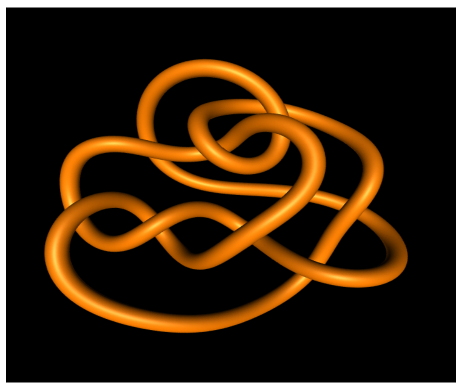
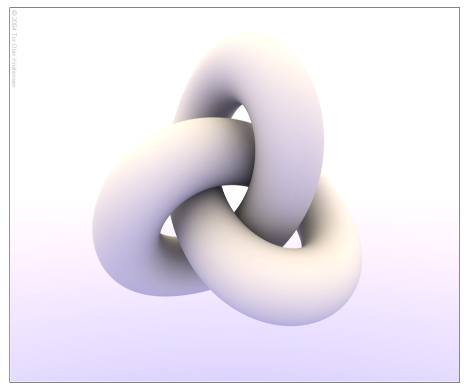
The main strategy will be to reduce questions about 3-colorings to linear algebra, via a certain incidence matrix associated to a knot diagram. Reidemeister moves give rise to Gauss operations (row or column operations) of this matrix. This is familiar from linear algebra, with the twist that our ground field is the finite field $\mathbb{F}_3$.

Questions to ask about the introductory section:

- Does it give the reader a good intuitive grasp of what the problem is?
- Does it express your approach to the problem?

1. Introduction (Third version, can serve as a model)

Mathematical knots are closed loops or strings in space, without kinks or selfintersections. The main question in knot theory is: *how can we tell whether a knot can be deformed into another one?* For instance, look at the following two knots (figures taken from [3,5]):

The one on the left is an *unknot*, which means that it can be deformed into a circle; in contrast, the one on the right, called a *trefoil knot*, is a genuine nontrivial knot. The first fact can be shown by going through an elementary sequence of transformations; however, the second one requires more advanced tools, namely *knot invariants*.

A knot invariant is a number (or other algebraic structure) attached to a knot, which is unchanged under deformations. Our project studies *knot 3-colorings*, which are ways of labeling segments of a knot with three possible labels. It will turn out that 3-colorability of a knot, and more generally the number of 3-colorings, is a knot invariant. This is the main result of the paper (Theorem 3.1), and will be shown using a case-by-case analysis of knot deformations. The unknot is not 3-colorable, but the trefoil is: hence, we can show that they cannot be deformed into each other (Example 4.2). Using the number of colorings as an invariant, we'll be able to find more inequivalent knots (Corollary 4.4).

Knot colorability can be generalized from 3 colors to n, for any $n \geq 3$. We discuss this generalization in Section 5. It turns out that these are genuinely better invariants: there are knots which can be distinguished by their 5-colorings but not by their 3-colorings (Example 5.2).

An (optional) background section *introduces the notions and notation* that you need throughout the paper. Suggestions:

- Watch out for material which is standard, but which you don't really use (or use only in spots).
- Watch out for possible misunderstandings.

2. Background (First version, already not bad)

Definition 2.1. An *oriented knot* is a smooth map $k : \mathbb{R} \to \mathbb{R}^3$ such that $k(s) = k(t)$ if and only if $s - t \in \mathbb{Z}$, as well as $k'(t) \neq 0$ for all t. An unoriented knot is such a map considered up to time-reversal $k(t) \leftrightarrow k(-t)$. In this paper, instead of k itself we consider the image $k(\mathbb{R}) \subset \mathbb{R}^3$, which is a closed loop.

Definition 2.2. Two oriented knots k_0, k_1 are called isotopic if there is a family of knots k_s, smooth with respect to the parameter $s \in [0, 1]$, joining them. Unoriented knots are called isotopic if, for some choice of orientation, the associated oriented knots are isotopic. The intuitive image is that the knot k_s moves around, without acquiring selfintersections or kinks. In fact, we will study knots via knot projections.

Definition 2.3. A knot projection is a smooth map $l : \mathbb{R} \to \mathbb{R}^2$ such that $l(t) = l(t + 1)$ for all t, $l'(t) \neq 0$ for all t, and with the at most finitely many double crossings. At each crossing, the two branches should meet transversally, and we distinguish one of them as overbranch (the other being the underbranch).

Questions to ask about the background section:

- Is the level of detail and precision appropriate?
- Is the exposition well-organized?

2. Background (Second version, more focused)

Intuitively, a *mathematical knot* $K \subset \mathbb{R}^3$ can be thought of as a closed, tangled loop of string in 3-dimensional space. The formal definition is that K is the image of an infinitely differentiable map $k : \mathbb{R} \to \mathbb{R}^3$ which satisfies:

- $k(t) = k(t+1)$ for all t (this makes K into a closed circle);
- $k'(t) \neq 0$ (this means that there are no kinks or singular points);
- $k(s) \neq k(t)$ for all $0 \leq s < t < 1$ (this avoids selfintersections).

However, we will mostly appeal to geometric intuition, avoiding formal details. Two knots are called *equivalent* if they can be deformed smoothly into each other.

Mostly, we will study knots through their projections. A knot projection is a circle drawn in the plane with only transverse selfintersections. Moreover, at each intersection, one of the two branches is distinguished as lying on top of the other.

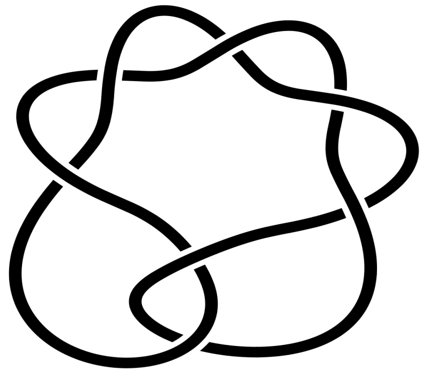
Questions to ask about the background section:

- Is the level of detail and precision appropriate?
- Is the exposition well-organized?

2. Background (Third version, more daring but appropriate)

A *knot* K is a circle smoothly embedded in three-dimensional space. Intuitively, it can be thought of as a closed piece of string. It can be tangled, but may not have kinks or selfintersections. Two knots are considered equivalent if they can be smoothly deformed into each other (again, without causing kinks or selfintersections). There are precise definitions of these notions involving calculus, but we will not really need them, since all our study of knots is done via their projections to the plane.

A *knot projection* P is a closed smooth curve in the plane $\mathbb{R}^2$, which has only ordinary selfintersections, and with some additional information. Having ordinary selfintersections means that at most two branches of the curve meet at any point. Moreover, the two branches which meet always cross each other transversally, meaning that they have different tangent directions. The additional information is that at any crossing point, we single out one of the branches as the overcrossing (and the other as the undercrossing). Knot projections are usually drawn like this:



Each knot projection P describes a knot $K = K(P)$, in a way which is unique up to equivalence. Different plane projections can also describe equivalent knots (for instance, in the Introduction we saw a very complicated projection whose associated knot is equivalent to the unknot, described by the circle in the plane).

There is no unique way of *presenting your results.* Questions to ask:

- Does the text express opinions clearly, and are those properly supported by evidence? (especially important if the results are not completely formalized)
- Are the proofs readable?

It's also important to distinguish between your own results and things you're learned by consulting the literature; we'll talk about that later.

2. Main theorem (first version, is it focused on the right thing?)

Theorem 3.1. 3-*colorability is a knot invariant.*

We will prove this by showing that it is invariant under Reidemeister moves. Reidemeister type I (Lemma 3.2) and II (Lemma 3.3) are relatively easy. Reidemeister III is harder and has to be divided into several sub-cases; we will only sketch that part (see the discussion at the end of the section).

Questions to ask about the presentation of your results:

- Does the text express opinions clearly, and are those properly supported by evidence?
- Are the proofs readable?

2. Main theorem (second version, much better)

Theorem 3.1. 3-*colorability is a knot invariant.*

We will prove this by showing that it is invariant under Reidemeister moves. Type I is straightforward. Type II is still simple, but already requires the distinction between two cases (of strands with equal or unequal colors). Type III is divided into many cases, and we'll only do the most interesting one (where the colors are as distinct as possible).

Questions to ask about the presentation of your results:

- Does the text express opinions clearly, and are those properly supported by evidence?
- Are the proofs readable?

3. Main theorem (third version, more daring and even better)

Theorem 3.1. 3-*colorability is a knot invariant.*

Our approach is slightly indirect, and involves the arithmetic of the field $\mathbb{F}_3$ of integers modulo 3. Given a nontrivial knot projection P with n crossings, we define an n by n matrix $A = A(P)$ with coefficients in $\mathbb{F}_3$, as follows. Columns of the matrix are labeled by arcs, and rows are labeled by crossings. The (i, j)-th entry A_{ij} is 1 if the j-th crossing lies on the i-th arc (which can mean that it is an overcrossing, or that it is an undercrossing and the arc ends there). All other entries are set to be 0. We will show first the following:

Lemma 3.2. *A vector $v \in (\mathbb{F}_3)^n$ is a coloring of P if and only if $Av = 0$.*

This includes the trivial colorings $v = (i, \dots, i)$, which always exist and are ruled out by our usual terminology. As an easy consequence, we get:

Lemma 3.3. *The knot projection P is colorable if and only if* $\operatorname{rank}(A) < n - 1$.

After that, we examine how the matrix changes under Reidemeister moves:

Lemma 3.4. *The effect of any Reidemeister move on the matrix A can be written as composition of the following operations, and their inverses:*

- *extending the matrix by adding a new column and row at the end, with the bottom right diagonal entry being* 1 *and all other new entries being* 0*;*
- *permuting rows or columns;*
- *Gauss operations (taking a multiple of some row and adding it to another row, and the same for columns).*

From linear algebra, we know that the rank of a matrix is not affected by such operations. Hence, Theorem 3.1 follows directly from Lemmas 3.3 and 3.4. The rest of this section contains the proof of the Lemmas.

Questions to ask about the presentation of your results:

- Does the text express opinions clearly, and are those properly supported by evidence?
- Are the proofs readable?

4. The number of colorings (first version, partial evidence only)

The result about 3-colorability can be strengthened as follows:

Experimental Fact 4.1. *The number of* 3*-colorings is a knot invariant.*

We have no rigorous proof of this. It is obviously true for unknots, since then the number of 3-colorings is always zero. Next, we have considered 20 different projections of the trefoil (Appendix 1). They all turn out to have the same number of colorings, supporting our conjecture. It is still possible that this is a special property of the trefoil, and fails for more complicated knots.

Experimental Fact 4.2. *The number of* 3*-colorings of a given knot projection is always of the form* $3^n - 3$*, for some integer* n*.*

We have taken the first 12 knots from the classical knot tables [7], and computed their numbers of colorings by hand. The result of the computation (see Appendix 2) supports Experimental Fact 4.2. We have actually found a general proof in the literature [4], but it uses algebra arguments which are quite different from the ones used here.

Questions to ask about the presentation of your results:

- Does the text express opinions clearly, and are those properly supported by evidence?
- Are the proofs readable?

4. The number of colorings (second version, better evidence)

3-colorability as a knot invariant has limited power, since it can at most divide all possible knots into two classes. We would like to refine it as follows:

Conjecture 4.1. *The number of* 3*-colorings is a knot invariant.*

The natural way to approach this is by inspection of the proof of Theorem 3.1. Under Reidemeister moves of type I and II, that proof provides a simple bijection between colorings of the original knots and of the modified one, which provides partial evidence for the conjecture. Reidemeister III is more complicated, and we could not complete the argument in all cases.

Conjecture 4.2. *The number of* 3*-colorings of a given knot projection is always of the form* $3^n - 3$*, for some integer* n*.*

$3^n - 3$ is always even and divisible by three (by computations modulo 2 and 3, respectively). Note that we can permute the colors in any 3-coloring, which means that the number of colorings is always divisible by 6. This provides partial evidence for the conjecture. One could of course also get experimental evidence by looking at the classical knot tables [7].

Questions to ask about the presentation of your results:

- Does the text express opinions clearly, and are those properly supported by evidence?
- Are the proofs readable?

4. The number of colorings (third version, even better organization)

In addition to the basic question of whether a knot is colorable, one can also look at the number of possible colorings.

Empirical Fact 4.1. *The number of* 3*-colorings is a knot invariant.*

Empirical Fact 4.2. *The number of* 3*-colorings is always of the form* $3^n - 3$*, for some integer* n*.*

The first fact is important since it yields an invariant that can have potentially infinitely many different values. The second fact restricts the range of this invariant. We cannot prove either statement rigorously, but there is partial evidence of various kinds:

Theoretical evidence. The number of 3-colorings is invariant under Reidemeister moves of type I and II. This can be seen by inspecting our proof of Theorem 3.1. Namely, if two knots are related by such a move, the argument from the proof provides bijection between their possible 3-colorings. If this argument could be extended to type III, it would provide a complete proof of Experimental Fact 4.1.

Experimental evidence. We have looked at 10 pairs of different projections of knots, differing from each other by a type III move. In all cases, the number of 3-colorings is the same before and after the move (Appendix 1), as predicted by Experimental Fact 4.1. We have also taken the first 12 knots from the classical knot tables [7], and computed their numbers of colorings by hand. The result of the computation (see Appendix 2) supports Experimental Fact 4.2.

You need to *cite all references* (from books to webpages) that you consulted and which had an impact on your report. Questions to ask about citations:

- Can the reader follow what results you're using and how?
- Can the reader see clearly where your original contribution lies?

Samples of what to avoid, if possible:

Useful probability textbooks are [1,2,15]...

We will use without further comment results from [5]...

Inspired by [3], we introduce the following matrix...

[5, Theorem 7] then yields the desired result...

The following argument is partly borrowed from [3] and partly original...

Good practice:

Knots will be listed in the notation from [5, pages 210-218]...

We apply the Central Limit Theorem [5, Theorem 15.7] to our probability distribution P, and conclude that...

Recall the main theorem about Reidemeister moves:

Theorem [4, Theorem 11]. *Let two knot ...*

We take the following definition from [3, pages 10–12], which treats only the case $n = 3$, and generalize it to all n.

Here are some *good practices in writing*.

- Early collaboration is important. If you don't understand your teammates' writing, chances are that no one else will, so insist until you get a good explanation. Try to keep the current draft accessible to the entire team at all times (via Dropbox or Google Docs). Do not hold on to your parts until they are finished!
- Revision is important. If a section is murky, it may be better start it again from scratch, organizing the material differently. Keep the main points and statements in mind as your principal goal.
- There is a 12-page limit. If you have significantly more, consider removing those parts that do not have clear goals or statements.

You will get feedback on the official draft (but somewhat less on the final version). Practice in writing mathematics will improve your general communications skills (and will provide you with samples of writing to show in job interviews).