ON AN EFFICIENT SQUARE-ROOT ALGORITHM

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In this paper, we prove the correctness of an algorithm for finding square roots.

**Theorem 1.** For any positive number \( \alpha \) and any \( x_1 > \sqrt{\alpha} \), we may define a sequence \( \{x_2, x_3, \ldots\} \) by the recursion formula

\[
x_{n+1} = \frac{1}{2} \left( x_n + \frac{\alpha}{x_n} \right).
\]

Then that \( \{x_n\} \) decreases monotonically and \( \lim_{n \to \infty} x_n = \sqrt{\alpha} \).

**Proof.** We prove this by two inductions.

Claim 1: \( x_n > \sqrt{\alpha} \) for all \( n \). To see this, note it holds for \( x_1 \), and if \( x_k > \sqrt{\alpha} \), then

\[
x_{k+1} - \sqrt{\alpha} = \frac{1}{2} \left( x_k + \frac{\alpha}{x_k} \right) - \sqrt{\alpha}
= \frac{1}{2} \left( x_k - \sqrt{\alpha} \right) \left( 1 - \frac{\sqrt{\alpha}}{x_k} \right)
> 0.
\]

Claim 2: \( x_{n+1} < x_n \) for every \( n \). To see this, first note if \( x_1 > \sqrt{\alpha} \), then

\[
x_2 = \frac{1}{2} \left( x_1 + \frac{\alpha}{x_1} \right)
< \frac{1}{2} \left( x_1 + \frac{\alpha}{\sqrt{\alpha}} \right)
= \frac{1}{2} \left( x_1 + \sqrt{\alpha} \right)
< x_1.
\]

If \( x_k < x_{k-1} \), then

\[
x_k - x_{k+1} = \frac{1}{2} \left( x_{k-1} - x_k + \frac{\alpha}{x_{k-1}} - \frac{\alpha}{x_k} \right)
= \frac{1}{2} \left( x_{k-1} - x_k \right) \left( 1 - \frac{\alpha}{x_k x_{k-1}} \right)
> 0,
\]

since \( x_k x_{k-1} > \alpha \) by Claim 1.

The sequence \( x_n \) is monotone decreasing and bounded below, therefore converges. To find the limit, note that \( x_{n+1} \) and \( x_n \) converge to the same thing, say \( x \). Then we have

\[
x = \frac{1}{2} \left( x + \frac{\alpha}{x} \right),
\]
or \( x = \sqrt{\alpha} \). \( \square \)
Theorem 2. Define \( \epsilon_n = x_n - \sqrt{\alpha} \). Then
\[
\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}.
\]
and so if we set \( \beta = 2\sqrt{\alpha} \) we have
\[
\epsilon_{n+1} < \beta \left( \frac{\epsilon_1}{\beta} \right)^{2^n}.
\]

Proof. It was show in the proof of Theorem 1 that
\[
x_{n+1} - \sqrt{\alpha} = \frac{1}{2} \left( x_n + \frac{\alpha}{x_n} \right) - \sqrt{\alpha}
\]
\[
= \frac{1}{2} \left( x_n - 2\sqrt{\alpha} + \frac{\alpha}{x_n} \right)
\]
\[
= \frac{1}{2} \left( x_n - \sqrt{\alpha} \right) + \left( -\sqrt{\alpha} + \frac{\alpha}{x_n} \right)
\]
\[
= \frac{1}{2} \left( x_n \left( 1 - \frac{\alpha}{x_n} \right) - \sqrt{\alpha} \left( 1 - \frac{\alpha}{x_n} \right) \right)
\]
\[
= \frac{1}{2} (x_n - \sqrt{\alpha}) \left( 1 - \frac{\sqrt{\alpha}}{x_n} \right)
\]
\[
= \frac{1}{2} \frac{(x_n - \sqrt{\alpha})^2}{x_n}
\]
\[
= \frac{\epsilon_n^2}{2x_n},
\]
and \( x_n > \sqrt{\alpha} \) for every \( n \), which gives the stated inequality. The rest follows by induction. \( \square \)