Scrambled Proof #2

Consider the following theorem and the “pieces” of the proof you’ve been given. Please put the pieces in logical order to form an understandable proof.

**Theorem:** Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $T$ be a bounded linear operator on $\mathcal{H}$. Then there exists a bounded linear operator $T^*$ on $\mathcal{H}$ such that, for all $x$ and $y$ in $\mathcal{H}$, $\langle Tx, y \rangle = \langle x, T^*y \rangle$.

**Proof:**
1. Finally, to show that $T^*$ is linear, a computation shows that for any complex numbers $\alpha$ and $\beta$ and $y_1$ and $y_2$ in $\mathcal{H}$, the vector $\alpha T^*(y_1) + \beta T^*(y_2)$ satisfies $\langle Tx, \alpha y_1 + \beta y_2 \rangle = \langle x, \alpha T^*(y_1) + \beta T^*(y_2) \rangle$ for all $x$, and using uniqueness, therefore, $T^*(\alpha y_1 + \beta y_2) = \alpha T^*(y_1) + \beta T^*(y_2)$ as required.

2. Therefore, $f_y$ is a bounded linear functional on $\mathcal{H}$ and since $\mathcal{H}$ is self-dual, there exists a unique vector $y^*$ in $\mathcal{H}$ such that $f_y(x) = \langle x, y^* \rangle$ for all $x$.

3. Thus, the candidate for $T^*$ is satisfactory, and the proof is complete.

4. Observe that this function $f_y$ is linear in $x$ by standard properties of the inner product and that it maps $\mathcal{H}$ to the complex numbers.

5. Next, observe that by construction, $T^*$ satisfies $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x$ and $y$ in $\mathcal{H}$.

6. We will construct the required $T^*$ vector by vector.

7. First, clearly $T^*$ so constructed maps $\mathcal{H}$ to $\mathcal{H}$.

8. Let $y$ be a fixed vector in $\mathcal{H}$ and consider the function $f_y$ defined on $\mathcal{H}$ by $f_y(x) = \langle Tx, y \rangle$ for all values of $x$.

9. Further, $f_y$ is bounded, since for any $x$, $|f_y(x)| = |\langle Tx, y \rangle| \leq \|T\||\|x\||\|y\|$.

10. Further, to show $T^*$ is bounded, note that $\|T^*y\| = \sup \{|\langle x, T^*y \rangle| : x \in \mathcal{H}, \|x\| \leq 1\} = \sup \{|\langle Tx, y \rangle| : x \in \mathcal{H}, \|x\| \leq 1\} \leq \sup \{||T||\|x\||\|y\| : x \in \mathcal{H}, \|x\| \leq 1\} = ||T||\|y\|$.

11. Let $\mathcal{H}$ and $T$ be as in the hypothesis.

12. Our candidate for $T^*$ is to set, for each $y$ in $\mathcal{H}$, $T^*(y) = y^*$, where $y^*$ is as constructed above for $y$. 
Key: 11, 6, 8, 4, 9, 2, 12, 5, 7, 10, 1, 3