Scrambled Proof #2

Consider the following theorem and the "pieces" of the proof you've been given. Please put the pieces in logical order to form an understandable proof.

Theorem: Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let T be a bounded linear operator on $\mathcal{H}$. Then there exists a bounded linear operator T^* on $\mathcal{H}$ such that, for all x and y in $\mathcal{H}$, $\langle Tx, y \rangle = \langle x, T^* y \rangle$.

Proof:

1. Finally, to show that T^* is linear, a computation shows that for any complex numbers α and β and y_1 and y_2 in $\mathcal{H}$, the vector $\alpha T^*(y_1) + \beta T^*(y_2)$ satisfies $\langle Tx, \alpha y_1 + \beta y_2\rangle = \langle x, \alpha T^*(y_1) + \beta T^*(y_2)\rangle$ for all x, and using uniqueness, therefore, $T^*(\alpha y_1 + \beta y_2) = \alpha T^*(y_1) + \beta T^*(y_2)$ as required.

2. Therefore, f_y is a bounded linear functional on $\mathcal{H}$ and since $\mathcal{H}$ is self-dual, there exists a unique vector y^* in $\mathcal{H}$ such that $f_y(x) = \langle x, y^*\rangle$ for all x.

3. Thus, the candidate for T^* is satisfactory, and the proof is complete.

4. Observe that this function f_y is linear in x by standard properties of the inner product and that it maps $\mathcal{H}$ to the complex numbers.

5. Next, observe that by construction, T^* satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all x and y in $\mathcal{H}$.

6. We will construct the required T^* vector by vector.

7. First, clearly T^* so constructed maps $\mathcal{H}$ to $\mathcal{H}$.

8. Let y be a fixed vector in $\mathcal{H}$ and consider the function f_y defined on $\mathcal{H}$ by $f_y(x) = \langle Tx, y\rangle$ for all values of x.

9. Further, f_y is bounded, since for any x, $|f_y(x)| = |\langle Tx, y\rangle| \leq \|T\|\|x\|\|y\|$.

10. Further, to show T^* is bounded, note that $\|T^*y\| = \sup\{|\langle x, T^*y\rangle| : x \in \mathcal{H}, \|x\| \leq 1\} = \sup\{|\langle Tx, y\rangle| : x \in \mathcal{H}, \|x\| \leq 1\} \leq \sup\{\|T\|\|x\|\|y\| : x \in \mathcal{H}, \|x\| \leq 1\} = \|T\|\|y\|$.

11. Let $\mathcal{H}$ and T be as in the hypothesis.

12. Our candidate for T^* is to set, for each y in $\mathcal{H}$, $T^*(y) = y^*$, where y^* is as constructed above for y.

Key: 11, 6, 8, 4, 9, 2, 12, 5, 7, 10, 1, 3