Cauchy Sequences and Convergence

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1 Introduction and definitions

In this paper, we present the proof of a theorem relating convergent sequences to Cauchy sequences. This theorem helps motivate the idea of a complete metric space and of the notion of topological completion. We begin by presenting the definitions we need to state the theorem.

Definition 1.1. *A sequence $(s_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is said to be a* Cauchy sequence *if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$ we have $d(s_n, s_m) < \varepsilon$.*

Definition 1.2. *A sequence $(s_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is said to* converge to *$L \in X$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $d(s_n, L) < \varepsilon$. If there exists $L \in X$ such that $(s_n)_{n\in\mathbb{N}}$ converges to L then we say that $(s_n)_{n\in\mathbb{N}}$* converges *or* is convergent.

With these definitions in hand, we proceed to our main result.

2 Main Theorem

Theorem 2.1. *Every convergent sequence $(s_n)_{n\in\mathbb{N}}$ in the metric space (X, d) is a Cauchy sequence.*

Proof. Suppose that $(s_n)_{n\in\mathbb{N}}$ is a convergent sequence. Then it has a limit L. By choosing a small ε, we have by the definition of convergence that eventually all terms of $(s_n)_{n\in\mathbb{N}}$ will be arbitrarily to L. Then choosing any two of these terms, say s_n and s_m, and applying the triangle inequality, we have that

$$d(s_m, s_n) \leq d(s_m, L) + d(L, s_n)$$

is also arbitrarily small. Then we have that $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, as desired. □

The key idea behind this proof is that the definitions of Cauchy sequence and convergent sequence differ primarily in that the latter explicitly references a limit while the former does not. In other words, we may think of a convergent sequence as one which "gets close to something" while a Cauchy sequence is one which "gets close together." Our intuition, formalized by the triangle inequality, shows that "getting close to something" necessarily means "getting close together," and this is our result.

Note that the converse of this statement is *not* true. In fact, motivated by this theorem, we define a special class of metric spaces (the topologically complete spaces) as those in which the converse does hold. Certain large, important classes of metric spaces are complete (see e.g. [R] Theorem 3.11).

References

[R] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition. McGraw-Hill, 1976.