# Are there gaps in the rationals?

There is a sense in which there is no empty space in the rational numbers: no matter how close together two rationals are, there are infinitely many other rationals between them. In this sense, there are no gaps in the rationals.

However, there is a sense in which there *is* empty space in the rational numbers. Let

$$A = \left\{r \in \mathbb{Q} : r > 0 \text{ and } r^2 \leq 2\right\} \quad \text{and} \quad B = \left\{s \in \mathbb{Q} : s > 0 \text{ and } s^2 \geq 2\right\}.$$

Clearly, $A$ is less than $B$. Nonetheless, there is no element $q \in \mathbb{Q}$ such that $q$ is between $A$ and $B$. Thus it is not always possible to find a rational number between any two *sets* of rationals. In this sense, there are gaps in the rationals.

To make matters geometrically explicit, consider the graph of $y = 2-x^2$ using only rational numbers. Notice that there's no $x \in \mathbb{Q}$ that makes $2 - x^2 = 0$. Thus, the graph never crosses the $x$-axis, despite the fact that the graph is above the $x$-axis at $x = 0$ and below the $x$-axis at $x = 3$. This demonstrates the existence of a more devios type of gap in the rationals, and the failure of $\mathbb{Q}$ to model our intuition about the physical world.