Are there gaps in the rationals?

There is a sense in which there is no empty space in the rational numbers: no matter how close together two rationals are, there are infinitely many other rationals between them. In this sense, there are no gaps in the rationals.

However, there is a sense in which there is empty space in the rational numbers. Let

\[ A = \{ r \in \mathbb{Q} : r > 0 \text{ and } r^2 \leq 2 \} \quad \text{and} \quad B = \{ s \in \mathbb{Q} : s > 0 \text{ and } s^2 \geq 2 \}. \]

Clearly, \( A \) is less than \( B \). Nonetheless, there is no element \( q \in \mathbb{Q} \) such that \( q \) is between \( A \) and \( B \). Thus it is not always possible to find a rational number between any two sets of rationals. In this sense, there are gaps in the rationals.

To make matters geometrically explicit, consider the graph of \( y = 2 - x^2 \) using only rational numbers. Notice that there’s no \( x \in \mathbb{Q} \) that makes \( 2 - x^2 = 0 \). Thus, the graph never crosses the \( x \)-axis, despite the fact that the graph is above the \( x \)-axis at \( x = 0 \) and below the \( x \)-axis at \( x = 3 \). This demonstrates the existence of a more devios type of gap in the rationals, and the failure of \( \mathbb{Q} \) to model our intuition about the physical world.