18.100C: Spring 2010

Recitation Worksheet: Proof by Contradiction

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Following are three theorems, each given with a proof that is constructed as a proof by contradiction. In each case, decide whether the proof structure can be changed to omit the contradiction. For those you feel can be proved with a direct argument avoiding contradiction, construct such a proof.

Theorem 1. Let (X, d) be a metric space, and let $E \subseteq X$. If x is a limit point of E, then in any neighbourhood N of x there are infinitely many points from E.

Proof: Assume, for a contradiction, that there exists an $r > 0$ such that the neighbourhood $B_r(x)$ contains only finitely many points from E. Thus $B_r(x)$ contains only finitely many points $e_1, \ldots, e_n$ in $E - \{x\}$. The number $s = \min\{r, d(e_1, x), \ldots, d(e_n, x)\}$ is strictly positive. Note that $e_j \notin B_s(x)$ for any $j \in \{1, \ldots, n\}$ since $d(x, e_j) \geq s$ for each such j. Also, as $s \leq r$, we have $B_s(x) \subseteq B_r(x)$; since $e_1, \ldots, e_n$ are the only points in $B_r(x) \cap (E - \{x\})$, we have shown that $B_s(x) \cap (E - \{x\})$ is empty. Hence, $B_s(x)$ is a neighbourhood of x that contains no points of $E - \{x\}$. But x is a limit point of E, so no such neighbourhood can exist. □

Theorem 2. There exists a subset E of $\mathbb{R}$ such that $\overline{E} = \mathbb{R}$ but $\overset{\circ}{E} = \varnothing$.

Proof: We suppose, for a contradiction, that no such set E exists. Consider the rational number $\mathbb{Q}$. For any $x \in \mathbb{R}$, and any $r > 0$, there is a rational number $q \in (x, x + r)$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. Thus, every neighbourhood $B_r(x)$ contains elements from $\mathbb{Q}$, which shows that x is a limit point of $\mathbb{Q}$. Since this is true for every real number x, it follows that $\overline{\mathbb{Q}} = \mathbb{R}$.
On the other hand, fix any $q \in \mathbb{Q}$ and any $r > 0$. There is an irrational number between q and $q + r$: if r is irrational, then $q + r/2$ is such a number; if r is rational and $r = m/n$ with $m, n \in \mathbb{N}$ then $q + \sqrt{m^2 + 1}/2n$ is such a number. Hence, the ball $B_r(q)$ is not contained in $\mathbb{Q}$. Since this is true for any $r > 0$, there is no neighbourhood of q contained in $\mathbb{Q}$, which means that q is not interior to $\mathbb{Q}$. Since this holds for every $q \in \mathbb{Q}$, it follows that the interior of $\mathbb{Q}$ is empty.
Thus, there exists a set (namely $E = \mathbb{Q}$) in $\mathbb{R}$ whose closure is $\mathbb{R}$ but whose interior is empty. This contradicts the assumption that no such E exists. □

Theorem 3. Let A be any set, and let $\mathscr{P}(A)$ denote the *power set* of A, the set of all subsets of A: $\mathscr{P}(A) = \{E\,;\, E \subseteq A\}$. There exists no surjection from A onto $\mathscr{P}(A)$.

Proof: Suppose, for a contradiction, that such a surjection $f\colon A \to \mathscr{P}(A)$ exists. Consider the subset $B \in \mathscr{P}(A)$ defined by $B = \{a \in A\,;\, a \notin f(a)\}$. We will demonstrate that, in fact, B is not in the image of f, contradicting the assumption that f is surjective.
Suppose, for a contradiction, that there exists some $x \in A$ for which $f(x) = B$. If $x \in B$, then by the definition of B, $x \notin f(x)$; but $f(x) = B$, and so this implies that $x \notin B$, contradicting the assumption that $x \in B$. Hence, we conclude that $x \notin B$, which means that $x \in f(x)$. But $f(x) = B$, so $x \in B$, contradicting the assumption that $x \notin B$. So the assumption that there is an x for which $f(x) = B$ must be false. It follows that B is not in the image of f. Therefore, the original assumption that a surjection $A \to \mathscr{P}(A)$ exists must be false. □