An Example of a Proof with Guiding Text

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...The last example I want to present has to do with permutations. A permutation of n is an ordering of the numbers from 1 to n. For example, a permutation of 7 is

$$1\ 4\ 3\ 7\ 2\ 6\ 5\,.$$

We will prove the following theorem:

Use formatting to emphasize the theorem statement and to clearly indicate the beginning of the proof.
If it would be helpful to readers, you may include explanatory text between the theorem statement and the proof.

Theorem 1. *Every permutation of n has either an increasing subsequence or a decreasing subsequence of length* $\lceil\sqrt{n}\rceil$.

A theorem statement must be precise, and should be worded generally enough so it can be applied in other contexts.
For example, "The permutation above has an increasing subsequence of length 3" is not suitable because it cannot be applied out of context.

Here $\lceil\cdot\rceil$ ("pronounced ceiling") means to round up to the next integer. For a permutation of 7, the theorem guarantees an increasing or a decreasing sequence of length at least $\lceil\sqrt{7}\rceil = 3$. The permutation above has an increasing subsequence 1, 4, 7 and a decreasing subsequence 4, 3, 2.

Proof of Theorem 1. We will associate to every number in the permutation an ordered pair of two integers. The first integer associated with a number k will be the length of the longest increasing subsequence ending with k, and the second will be the length of the longest decreasing subsequence ending with k. Thus, for the subsequence above, the ordered pairs will be

$$\begin{array}{ccccccc} 1 & 4 & 3 & 7 & 2 & 6 & 5 \\ \begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}2\\1\end{pmatrix} & \begin{pmatrix}2\\2\end{pmatrix} & \begin{pmatrix}3\\1\end{pmatrix} & \begin{pmatrix}2\\3\end{pmatrix} & \begin{pmatrix}3\\2\end{pmatrix} & \begin{pmatrix}3\\3\end{pmatrix} \end{array}.$$

Here, the ordered pair associated with 2 is $(2, 3)$ because the longest increasing subsequence ending with 2 is 1, 2, which has length 2, and the longest decreasing subsequence is 4, 3, 2, which has length 3.

Claim. All these ordered pairs are distinct.

"Guiding text" is text that guides the readers though the content by communicating the structure &/or purpose of the text to the readers.

First, we use the pigeonhole principle to show how this claim implies our theorem, and then we will prove this claim.

The pigeons will be the numbers 1, 2, . . ., n, of the permutation, and the holes will be the ordered pairs. Let ℓ_i be the length of the longest increasing subsequence of our permutation, and let ℓ_d be the length of the longest decreasing subsequence. It is clear that all the ordered pairs are less than or equal to (ℓ_i, ℓ_d), so the number of pairs is at most $\ell_i \ell_d$.

Explicitly identifying the pigeons and holes helps readers see how the principle is applied.

If we assume the claim that each pigeon fits into a different hole, then we have that the number of pigeons is at most the number of holes. There are n pigeons and $\ell_i \ell_d$ holes. This gives us

$$n \leq \ell_i \ell_d,$$

showing that either $\ell_i \geq \sqrt{n}$ or $\ell_d \geq \sqrt{n}$. Since ℓ_i and ℓ_d are integers, we can strengthen this to $\ell_i \geq \lceil \sqrt{n} \rceil$ or $\ell_d \geq \lceil \sqrt{n} \rceil$. $\square$

You can also use headings like this one to help readers follow the structure of the text.

Proof of Claim. Suppose that we have two numbers in our permutation s and t, with s coming before t, and let (a, b) and (c, d) be the ordered pairs associated with them.

$$\begin{array}{ccccc} \cdots & s & \cdots & t & \cdots \\ \cdots & \begin{pmatrix} a \\ b \end{pmatrix} & \cdots & \begin{pmatrix} c \\ d \end{pmatrix} & \cdots \end{array}$$

There is an increasing sequence of length a that ends with s. If $s < t$, we can add t to this increasing sequence to obtain an increasing sequence of length $a + 1$ that ends with t, showing that $c \geq a + 1$. A similar argument shows that if $s > t$, then $d \geq b + 1$. Thus, the two ordered pairs associated with s and t are distinct. This proves the claim. $\square$

This box signals the end of the proof. You could instead use "QED" or guiding text like "This proves the claim."

It is a fairly easy exercise to find permutations of n that have no increasing or decreasing sequences longer than $\lceil \sqrt{n} \rceil$ for any n, showing that our theorem is the strongest possible result.