of vector fields $X_{v(a)}$ parameterized by $a \in [0,1]$ lifts to a universal vector field $X_{v(s)}$ in the product $([0,1] \times F)$ and it descends to the quotient W.

Now on the conical end C we have:

$$\omega = \Theta + \pi^* dq = d\Theta_F + v \wedge ds + dr_S \wedge ds$$

because $\beta^* d\Theta_F = d\Theta_F$

and

$$dH = d(\pi_1 \circ H_F) + k'(r_S) dr_S$$

Hence, the Hamiltonian vector field of H is:

$$X_H = X_{H_F} - k'(r_S)\frac{\partial}{\partial s} + k'(r_S) X_{v(s)} - X_{H_F}(v)\frac{\partial}{\partial r_S}$$

Finally the action of an orbit γ of H in C is given by:

$$A_H(\gamma) = \int_\gamma \{(-X_{H_F} - k'(r_S)X_{v(s)})\Theta + r_S k'(r_S)\} - \pi_1^* H_F - k(r_S)$$

$\square$

Theorem 3.0.11. *Let (E,π) be a compact convex Lefschetz fibration in standard form. Let $H_p : \widehat{E} \to \mathbb{R}$ be Lefschetz admissible for E with slope p on the base and the fibre. Remember that $\widehat{E}$ has a convex symplectic structure $(\widehat{E}, \omega_E, \theta_E, \lambda_E, \phi_E)$.*

Then there is a cofinal family of Hamiltonians K_p with respect to the above convex symplectic structure such that:

1. *The periodic orbits of K_p of positive action are in 1-1 correspondence with the periodic orbits of H_p. This correspondance preserves index. Also the moduli spaces of Floer trajectories are the same between respective orbits.*
2. *$K_p \leq 0$ on $E \subset \widehat{E}$.*
3. *$K_p \to 0$ pointwise on E.*
4. *K_p is C^2 small in $E \subset \widehat{E}$.*

Now this theorem means that:

$$\varinjlim_p SH_*^{[0,\infty)}(K_p) = \varinjlim_p SH_*(H_p) \tag{1}$$

$SH_*^{[0,\infty)}(K_p) := SH_*(K_p)/SH_*^{(-\infty,0)}$ where $SH_*^{(-\infty,0)}$ is the symplectic homology group generated by orbits of negative action. We also have:

$$\varinjlim_p SH_*(K_p) = \varinjlim_p SH_*^{[0,\infty)}(K_p) \tag{2}$$

This is because there exists a cofinal family of Hamiltonians G_p such that:

1. $G_p \leq 0$ on $E \subset \widehat{E}$.
2. $G_p \to 0$ pointwise on E.
3. G_p is C^2 small in $E \subset \widehat{E}$.

(4) All the periodic orbits of G_p have positive action.

This means that there exist sequences p_i and q_i such that:

$$K_{p_i} \leq G_{q_i} \leq H_{p_{i+1}}$$

for all i. Hence:

$$\varinjlim_p SH_*^{[0,\infty)}(G_p) = \varinjlim_p SH_*^{[0,\infty)}(K_p)$$

Property 4 implies:

$$\varinjlim_p SH_*^{[0,\infty)}(G_p) = \varinjlim_p SH_*(G_p)$$

This gives us equation 2. Combining this with equation 1 gives:

$$\varinjlim_p SH_*(K_p) = \varinjlim_p SH_*(H_p)$$

This proves Theorem 1.6.2. Note: the Lefschetz fibration in Theorem 1.6.2 may not be in standard form, but we can deform it using Lemma 3.0.6 to a Lefschetz fibration in standard form. This induces an isomorphism between respective symplectic homology groups associated to each Lefschetz fibration.

Proof. of Theorem 3.0.11

We will slightly modify the proof of a related result in [14].

Also, we will use the notation set up already in 3.0.10.

We assume that the period spectra of ∂F and ∂S are discrete and injective.

The Hamiltonians H_F and H_S have slope $\lambda \notin \mathcal{S}(S) \cup \mathcal{S}(F)$. We also assume that H_F and H_S have this slope outside a small neighbourhood of F and S respectively. We will assume that H_F and H_S are C^2 small on the interior of F and S respectively.

What we want to do is to choose some convex symplectic structure

$$(E, \omega_E, \theta_E, \lambda_E, \phi_E)$$

on $\widehat{E}$ and a Hamiltonian H_3 so that there exist constants c_1, c_2, ϵ such that:

(1) $H_3 = H$ on E.
(2) Any curve in $\widehat{E}$ with each end converging to an orbit in E satisfying a Floer type equation (e.g Floer trajectory or pair of pants) is entirely contained in E.
(3) on $\{\phi_E \in [c_1, c_2]\}$ we have that H_1 is constant.
(4) on $\phi_E > c_2 + \epsilon$ we have that H_1 is linear with respect to the conical end of this convex symplectic structure.
(5) Any additional orbits (i.e orbits outside E) have negative action.

We will acheive this in 4 sections $(a) - (d)$.

Note H_3 will be constructed in 3 stages in sections (a),(b),(c) respectively (i.e we first construct H_1 from H in (a) and then H_2 from H_1 in (b) and then H_3 from H_2 in (c)).

In section (a) we will construct a Hamiltonian $H_{F,1}$ so that:

(1) on F, $H_{F,1}$ is equal to H_F.
(2) on $r_F \geq A$, $H_{F,1}$ is constant for some A to be defined later.
(3) $H_{F,1}$ is a function of r_F on the conical end of F.

We also construct a similar Hamiltonian $H_{S,1}$ which is associated with H_S. Finally in this section, we show that the orbits of

$$H_1 := \pi^*(H_{S,1}) + \pi_1^*(H_{F,1})$$

outside E have negative action. We already know that the orbits inside E are the same as the orbits of H because $H = H_1$ inside E.

In section (b) we will construct a Hamiltonian H_2 such that:

(1) $H_2 = H_1$ on $r_S \leq A, r_F \leq A$.
(2) H_2 is constant outside $r_S \leq B, r_F \leq B$ for some constant $B > A$.
(3) Any orbit of H_2 outside $r_S \leq A, r_F \leq A$ has negative action. This ensures that all the orbits of H_2 of positive action are the same as the orbits of H.

In section (c) we will finally construct H_3. We choose some admissible Hamiltonian K with respect to the convex symplectic structure

$$(E, \omega_E, \theta_E, \lambda_E, \phi_E)$$

which is equal to 0 on $r_S \leq C, r_F \leq C$ for some chosen $C > B$. Then we let $H_3 := H_2 + K$. We also ensure that K has slope proportional to $\sqrt{\lambda}$ which ensures that the additional orbits created on top of the orbits of H_2 have negative action.

In section (d) we will show that no Floer trajectory of H_3 connecting orbits inside E can intersect $r_F = C$ or $r_S = C$. If we combine this fact with the maximum principle in Lemma 3.0.5 and also a maximum principle from [13, Lemma 1.5] we find that any Floer trajectory connecting orbits inside E must be contained in E. This ensures that the Floer trajectories connecting orbits inside E are identical to the Floer trajectories of H and hence we get that:

$$SH_*^{[0,\infty)}(H_3) = SH_*(H)$$

And this gives us our result.

Define:

$$\mu_\lambda := \operatorname{dist}(\lambda, \mathcal{S}(S) \cup \mathcal{S}(F))$$

(a) We first modify a construction due to Herman in [10] which takes some normal cofinal Hamiltonian on a finite type convex symplectic manifold and makes it constant near infinity so that the only added periodic orbits have negative action. We need to modify this argument because we need greater control over the Hamiltonian flow $X_{\pi^* H_S}$.

From now on we will assume that $H_S = 0$ on S and is equal to $k(r_S)$ for $r_S \geq 1$. Similarly we assume that $H_F = 0$ on F and is a function of r_F on $r_F \geq 1$.

The first thing we need to do is to modify H_S and H_F to $H_{S,1} : S \to \mathbb{R}$ and $H_{F,1} : F \to \mathbb{R}$ so that they are constant at infinity and such that the additional orbits added to $H_1 := H_{F,1} + \pi^* H_{S,1}$ have negative action.

We will use all the notation as in the proof of 3.0.10.

Define:

$$R^s := \sup |X_{v(s)}(\Theta_F)|$$
$$R := \sup\{R^s : s \in [0,1]\}$$

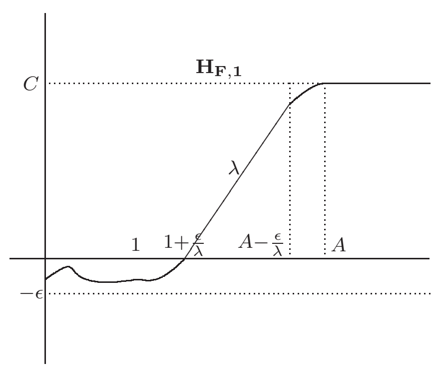
Define:

$$A = A(\lambda) := (6+R)\lambda/\mu_\lambda > 1$$

We can assume that $A > 1$ because we can choose μ_λ to be arbitrarily small.

We define $H_{F,1}$ to be equal to H_F on $r_F < A - \frac{\epsilon}{\lambda}$. Hence on the interior of F, $H_{F,1}$ has C^2 norm $\leq \epsilon$. Set $H_{F,1} = h_F(r_F)$ for $r_F \geq 1$ with non negative derivative. $h'_F(r_F)$ is equal to λ on $[1+\frac{\epsilon}{\lambda}, A - \frac{\epsilon}{\lambda}]$ For $r_F \geq A$ set $h_F(r_F)$ to be constant and equal to C where C is arbitrarily close to $\lambda(A-1)$. $H_{F,1}$ takes values in $[-\epsilon, \epsilon]$ for $r_F \in [1, 1+\frac{\epsilon}{\lambda}]$ and in $[\lambda(A-1) - 2\epsilon, \lambda(A-1)]$ for $r_F \geq A - \frac{\epsilon}{\lambda}$. Here is a picture:

Figure 3.0.12.



For notational convenience we will write $H_{F,1}$ instead of $\pi_1^* H_{F,1}$.

Assume that $H_{S,1}$ is a Hamiltonian such that on the conical end C we have that $H_{S,1}$ is equal to $k(r_S)$. We want $H_{S,1}$ to behave in a similar way to $H_{F,1}$. (i.e we have that the graph of $k(r_S)$ is the same as the graph in figure 3.0.12).

We want to show that the additional orbits of $H_1 := H_{F,1} + \pi^* H_{S,1}$ only have negative action. These additional orbits lie in the region $r_S \in (A - \frac{\epsilon}{\lambda}, A)$ and $r_F \in (A - \frac{\epsilon}{\lambda}, A)$.

We will first consider the orbits in $r_S \in (A - \frac{\epsilon}{\lambda}, A)$. Now, orbits of $H_{F,1}$ have action at most λ because $h'_F \leq \lambda$, i.e. $\int_{\text{orbit}} -X_{H_{F,1}}\Theta_F \leq \lambda$.

Let p be a point on some orbit o. Remember that the smallest distance between λ and the period spectrum of ∂F is $\geq \mu_\lambda$. Hence near p we have $|k'(r_S)| < \lambda - \mu_\lambda$. Hence $|k'(r_S)X_{v(s)}\Theta| \leq R(\lambda - \mu_\lambda)$ and $r_S k'(r_S) \leq A(\lambda - \mu_\lambda)$.

Also, because $|H_{F,1}v|_{C^2}$ is smaller than ϵ, we have that the orbit cannot move more than ϵ away from $r_S = A$. Hence the action of an orbit near $r_S = A$ is less than or equal to:

$$\lambda + (R + A)(\lambda - \mu_\lambda) - C + 2\epsilon$$

$$\leq (R + 1 + 1 + A - A - (6 + R))\lambda + 2\epsilon \leq -3\lambda \to -\infty$$

Now the case for orbits near $r_F = A$ is exactly the same as in Oancea's paper [14]. Near $r_F = A$ we have that $v = 0$, hence the action is at most:

$$\lambda + A(\lambda - \mu_\lambda) - C + \epsilon$$

$$\leq (1 + A - A + 1 - (6 + R))\lambda + \epsilon \leq -3\lambda \to -\infty$$

Hence all the additional orbits of H_1 have actions tending to $-\infty$.

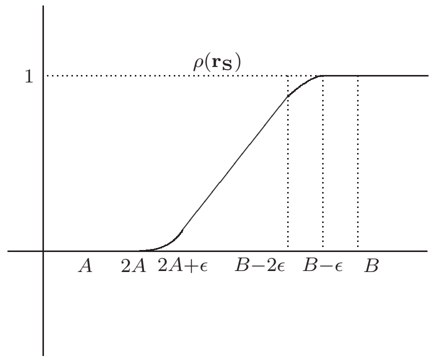
(b) Now we modify H_1 so that it is constant and equal to $2C$ outside the compact set $\{r_S \leq B, r_F \leq B\}$ with $B = A\sqrt{\lambda}$. This is true already on $\{r_S \geq A\} \cap \{r_F \geq A\}$, so we only need to consider the case $\{r_S \geq A\} \cap \{r_F \leq A\}$ and $\{r_F \geq A\} \cap \{r_S \leq A\}$. Now the case $\{r_F \geq A\} \cap \{r_S \leq A\}$ is exactly the same as the case Oancea dealt with in [14, section (c)]. (Note: in Oancea's paper, $A = 5\lambda/\mu_\lambda$ instead of $(6 + R)\lambda/\mu_\lambda$ but this makes no difference.) In Oancea's paper he deals with this case by modifying $\pi_1^* H_{F,1}$ to some new Hamiltonian $H_{F,2}$.

We will mimick Oancea's paper for the case $\{r_S \geq A, r_F \leq A\}$. This will involve modifying the Hamiltonian $\pi^* H_S$ to some new Hamiltonian $H_{S,2}$. Let:

$$H_{S,2} : W \times [A, \infty) \longrightarrow \mathbb{R}$$

$$H_{S,2}(x, s, r_S) = (1 - \rho(r_S))H_{F,1}(x) + \rho(r_S)C$$

where x is a point in F and s parametrizes $[0, 1]$. Also, $\rho : [A, \infty) \to [0, 1]$ with $\rho = 0$ on $[A, 2A]$, $\rho = 1$ for $r_S \geq B - \epsilon$, ρ strictly increasing on $[2A, B - \epsilon]$, and $\rho' = \text{const} \in \left[\frac{1}{B-2A-\epsilon}, \frac{1}{B-2A-3\epsilon}\right]$ on $[2A + \epsilon, B - 2\epsilon]$. The graph of ρ is:



We also have:

$$dH_{S,2} = (1-\rho(r_S))dH_{F,1} + (C - H_{F,1})\rho'(r_S)dr_S$$

$$X_{H_{S,2}} = (1-\rho(r_S))(X_{H_{F,1}} - X_{H_{F,1}}(v)\frac{\partial}{\partial r_S}) + (C - H_{F,1})\rho'(r_S)(X_{v(s)} - \frac{\partial}{\partial s})$$

Let $H_2 := H_{S,2} + H_{F,2}$.

We have assumed earlier that $X_{H_F}(v) = 0$, and hence $X_{H_{F,1}}(v) = 0$. This means that projecting orbits down to the base S produces orbits of the Hamiltonian H_S. In particular we can assume that the orbits of H_2 on $r_S \geq 1$ stay in each level set $r_S = \text{const}$.

For some orbit o of H_2, let:

$$A_1 := -\int_o \left[(1-\rho(r_S))(X_{H_{F,1}}) + (C - H_{F,1})\rho'(r_S)(X_{v(s)})\right](\Theta)$$

$$A_2 := \int_o \left[(C - H_{F,1})\rho'(r_S)r_S\right]$$

The action of this orbit o is equal to:

$$A_1 + A_2 - (C - H_{F,1})\rho(r_S) - C$$

(Remember $H_{F,2} = C$ on $\{r_S \geq A, r_F \leq A\}$)

We first consider orbits where $v \neq 0$ on some part of the orbit. Now these orbits are located in the interior of each fiber F. Hence, we can assume that H_F is negative and is C^2 bounded by ϵ. Also we may assume that $X_{H_{F,1}}(\Theta)$ is bounded above by ϵ. Now, because $X_{H_{F,1}}(v) = 0$, the r_S coordinate of the orbit is constant, hence we only need to consider 3 cases (i,ii,iii) for these orbits:

(i) $r_S \in [A, 2A] \cup [B - \frac{\epsilon}{\lambda}, \infty)$

Now, $\rho' = 0$ and $X_{H_{F,1}}\Theta$ is bounded above by ϵ. Hence the action is bounded above by $\epsilon - C$.

(ii) $r_S \in [2A, \frac{A+B}{2}]$

$\rho' \leq \frac{1}{B-2A-3\epsilon}$. S is bounded above by $\frac{A+B}{2}+1$. Also, $|X_{v(s)}(\Theta)|$ is bounded above by the constant R. Now, for large enough λ we also have that $\frac{A+B}{2}\frac{1}{B-2A-3\epsilon}$ is bounded above by $\frac{3}{4}$ because this expression tends to $\frac{1}{2}$ as $\lambda \to \infty$. Also, we can ensure that $\epsilon + C.\frac{1}{B-2A-3\epsilon}.R \leq \frac{1}{8}C$ for large enough λ. Hence our action is bounded above by:

$$\epsilon + C.\frac{1}{B-2A-3\epsilon}.R + (C-\epsilon).\frac{A+B}{2}\frac{1}{B-2A-3\epsilon} - C$$
$$\leq -\frac{1}{8}C$$

for large enough λ.

(iii) $r_S \in [\frac{A+B}{2}, B-\frac{\epsilon}{\lambda}]$

In this case we have $\rho \in [\frac{1}{2}, 1]$. Hence for λ big enough we have that the action is bounded above by:

$$\epsilon + C.\frac{1}{B-2A-3\epsilon}.R + (C-\epsilon).\frac{1}{B-2A-3\epsilon}.(B-\frac{\epsilon}{\lambda}) - (C-\epsilon).\frac{1}{2} - C$$
$$\leq -\frac{1}{8}C$$

Hence all orbits which pass through $v \neq 0$ have negative action in $W \times [A, \infty)$. Now, when $v = 0$ the action of the orbits are the same as in Oancea's paper [14] (although $A = 5\lambda/\mu_\lambda$ instead of $(6+R)\lambda/\mu_\lambda$, but this doesn't matter). Hence, these orbits also have action tending to $-\infty$ as well.

Hence we have a Hamiltonian which is equal to H on E and is constant and equal to $2C$ further out, and such that the only additional orbits have negative action.

(c) Finally we need to make this Hamiltonian cofinal by choosing some contact boundary and forcing H to be linear at this contact boundary, and such that the only additional orbits have negative action as well.

Let Z be the Liouville vector field which is ω-dual to $\theta := \Theta + \pi^* q$. Then this vector field is expressed as:

$$Z := Z' + (r_S - Z'(v))(\frac{\partial}{\partial r_S})$$

where Z' is the Liouville vector field in F associated to $\Theta|_{\pi^{-1}(y)}$. We assume that λ is big so that $A\sqrt{\lambda} = B > |Z'(v)|$. Consider the sets:

$$\mathbf{I} = \partial S \times [1, \infty) \times \partial F \times [1, \infty)$$
$$\mathbf{II} = S \times \partial F \times [1, \infty)$$
$$\mathbf{III} = W \times [1, \infty)$$

(see Oancea's paper: [14, figure 3]).

We define a hypersurface $\Sigma \subset \hat{E}$ such that:

$$r_S|_{\Sigma \cup \mathbf{III}} = \alpha > 1$$
$$r_S|_{\Sigma \cup \mathbf{I}} \in [1, \alpha]$$

$$r_F|_{\Sigma \cup \mathbf{II}} = \beta > 1$$
$$r_F|_{\Sigma \cup \mathbf{I}} \in [1, \beta]$$

We can ensure that Z is transverse to this hypersurface, and hence the flow of Z gives us a map:

$$\Psi : \Sigma \times [1, \infty) \to \hat{E}$$

which gives us a conical end for $\hat{E}$. Let r be the coordinate for the $[1, \infty)$ part of $\Sigma \times [1, \infty)$. Then $\Psi^{-1}(\{r_S \geq B\} \cup \{r_F \geq B\}) \supset \{S \geq B\}$

H_2 is constant and equal to $2C$ on $\{r_S \geq B\} \cup \{r_F \geq B\}$

Let K be a Hamiltonian which is equal to 0 on the region $r < B + \epsilon$ bounded by Σ and is equal to $l(r)$ outside $r \geq B + \epsilon$ where $l'(r) \geq 0$ and for $r \geq B + \epsilon$ we have $l'(r) = \mu \notin \mathcal{S}(\Sigma)$, where μ will be arbitrarily close to $L\sqrt{\lambda}$ where L is some constant chosen later. The point is that $K = 0$ on the region

$$\{r_S \leq B\} \cap \{r_F \leq B\}$$

This means that the orbits lie in the region where H_3 is constant and equal to $2C$.

Define:

$$H_3 := H_2 + K$$

Now the actions of the orbits of K are bounded above by $BVL\sqrt{\lambda}$ for some constant V. Choose $L < \frac{1}{V}$. Hence the orbits of H_3 inside $r \geq B + \epsilon$ have action bounded above by:

$$(B + \epsilon)\sqrt{\lambda} - 2C = (\sqrt{\lambda}A + \epsilon)\sqrt{\lambda} - \lambda(A - 1)$$

For large enough λ we have that this quantity is negative. Hence the actions of the additional orbits are negative.

(d) Finally using [14, Lemma 1], we have that any curve u passing through $\{r_S \in [A, 2A]\}$ must have area greater than cA for some constant c (i.e $\pi \circ u$ has area less than the area of u, so we can use [14, Lemma 1]). Now the actions of orbits inside E are bounded above by $P\lambda$ where P is some constant. This means that for large enough λ (i.e so that $P\lambda < cA$) we have ensured that no Floer trajectory between orbits of positive action can pass through $\{r_S \in [A, 2A]\}$. We have a similar statement for r_F.

Hence by the maximum principle (cf. Lemma 3.0.5 and [13, Lemma 1.5]) we have that any Floer trajectory connecting orbits of positive action stay within $\{r_S \leq 1, r_F \leq 1\}$ (this uses the fact that on $\{r_F \leq 2A\} \cap \{r_S \leq 2A\}$ we have that our Hamiltonian H_3 is equal to $H_1 = \pi^* H_{S,1} + \pi_1^* H_{F,1}$).

Note: Lemma 1 requires that the Hamiltonian be equal to 0 on $\{r_S \in [A, 2A]\}$ which means that it cannot have non-degenerate orbits. This problem can be solved as follows: Let H_k be a sequence of Hamiltonians with non-degenerate orbits and let J_k be a seqence of complex structures such that (H_k, J_k) C^2 converges to (H, J) as $k \to \infty$. If there is a Floer trajectory passing through $\{r_S \in [A, 2A]\}$ for some sequence of (H_k, J_k)'s converging to (H, J) then by Gromov compactness (see [1]) we have that there is a Floer trajectory of H passing through $\{r_S \in [A, 2A]\}$. But this is impossible, hence for some large enough k we have no Floer trajectory passing through $\{r_S \in [A, 2A]\}$.

Note, we can use an identical argument with the pair of pants surface satisfying Floer type equations.

□

3.1. **A better cofinal family for the Lefschetz fibration.** In this section we will prove Theorem 1.6.4.

We consider a compact convex Lefschetz fibration (E, π) fibred over the disc (i.e. $S = \mathbb{D}$). Basically the cofinal family is such that $H_F = 0$. This means that the boundary of F does not contribute to the symplectic homology of the Lefschetz fibration. The key idea is that near the boundary of F the Lefschetz fibration looks like a product $\mathbb{D} \times \mathrm{nhd}(\partial F)$ and because the symplectic homology of the disc is 0 we should get that the boundary contributes nothing.

Statement of Theorem 1.6.4:

If $S = \mathbb{D}$, the unit disc, then

$$SH_*(E) \cong SH_*^{\mathrm{lef}}(E)$$

From now on we will use the same notation as established in the proof of lemma 3.0.10.

Before we prove Theorem 1.6.4, we will write a short lemma on the $\mathbb{Z}$ grading of $SH_*(E)$.

Lemma 3.1.1. *Let E be a convex Lefschetz fibration with base S and a smooth fibre $F = \pi^{-1}(a)$ $(a \in S)$. Suppose we have trivialisations of $\mathcal{K}_{\widehat{E}}$ and $\mathcal{K}_{\widehat{S}}$ (these are the canonical bundles for $\widehat{E}$ and $\widehat{S}$ respectively); these naturally induce a trivialisation of $\mathcal{K}_{\widehat{F}}$ away from F. If we smoothly move a, then this smoothly changes the trivialisation.*

Proof. of Lemma 3.1.1. We choose a $J \in \mathcal{J}^h(E)$. The bundle E away from E^{crit} has a connection induced by the symplectic structure. Let A be defined as in 1.6.1. Let U be a subset of A where

(1) π is J holomorphic.
(2) $A \setminus U$ is relatively compact in E.
(3) U is of the form $r \geq K$ where r is the coordinate for $[1, \infty)$ in A (see definition 1.6.1).

This means that in U, we have that the horizontal plane bundle H is J holomorphic. Choose a global holomorphic section of $\mathcal{K}_{\widehat{S}}$ and lift this to a section s of H. Choose a global holomorphic section t of $\mathcal{K}_{\widehat{E}}$. Now the tangent bundle of $\widehat{F}$ is isomorphic to the ω orthogonal bundle T of H. This is also a holomorphic bundle. There exists a unique holomorphic section w of X such that $s \wedge w = t$. Hence, w is our nontrivial holomorphic section of T in $U \cup \widehat{F}$.