Definition 5.3. *Let M be a manifold with contact form α. Let* $S : \{\text{Reeb orbits}\} \to \mathbb{R}$, $S(o) := \int_o \alpha$. *Then the* **period spectrum** $\mathcal{S}(M)$ *is the set* $\operatorname{im}(S) \subset \mathbb{R}$. *We say that the period spectrum is discrete and injective if the map S is injective and the period spectrum is discrete in $\mathbb{R}$.*

Definition 5.4. *Let H be a Hamiltonian on a symplectic manifold M. Then the* **action spectrum** $\mathcal{S}(H)$ *of H is defined to be:*

$$\mathcal{S}(H) := \{A_H(o) : o \text{ is a 1-periodic orbit of } X_H\}.$$

A_H is the action defined in section 2.3.

We let F be a smooth fibre of (E, π) and $\Theta_F := \Theta|_F$. Also we let S be the base of this fibration. Let r_S and r_F be the "cylindrical" coordinates on $\hat{S}$ and $\hat{F}$ respectively (i.e. $\omega_S = d(r_S\theta_S)$ on the cylindrical end at infinity and similarly with r_F). Let W be some connected component of the boundary of S. Let $C := \pi^{-1}(W) \times [1, \infty)$. Note: we will sometimes write r_S instead of $\pi^* r_S$ so that calculations are not so cluttered. We hope that this will make things easier to understand for the reader.

The boundary of E is a union of 2 manifolds whose boundaries meet at a codimension 2 corner. We can smooth out this corner so that E becomes a compact convex symplectic manifold M such that the completion $\widehat{M}$ is exact symplectomorphic to $\widehat{E}$. This means we can view M as an exact submanifold of $\widehat{E}$. We will let $\partial M \times [1, \infty)$ be the cylindrical end of $\widehat{E} = \widehat{M}$ and we will let r be the coordinate for the interval $[1, \infty)$. We will assume that the period spectrum of ∂M is discrete and injective. Let $\varrho_p : \widehat{E} \to \mathbb{R}$ be an admissible Hamiltonian on $\widehat{M} = \widehat{E}$ with slope p with respect to the cylindrical end $\partial M \times [1, \infty)$ where p is a positive integer. We will also assume that $\varrho_p < 0$ inside M and that ϱ_p tends to 0 in the C^2 norm inside M as p tends to infinity, and that $\varrho_p = h_p(r)$ in the cylindrical end. We assume that $h_p'(r) \geq 0$ for all r and $h_p'(r) = p$ for $r \geq 2$. We also assume that $h_p''(r) \geq 0$ for all r. We can perturb the boundary of M to ensure that no positive integer is in the period spectrum of ∂M and hence p is not in the action spectrum. Hence the family $(\varrho_p)_{p \in \mathbb{N}_+}$ is a cofinal family of admissible Hamiltonians.

Theorem 5.5. *There is a cofinal family of Lefschetz admissible Hamiltonians $K_p : \widehat{E} \to \mathbb{R}$ and a family of almost complex structures $J_p \in \mathcal{J}_{\text{reg}}(\widehat{E}, K_p)$ such that for $p \gg 0$:*

1. *The periodic orbits of K_p of positive action are in 1-1 correspondence with the periodic orbits of ϱ_p. This correspondence preserves index. Also the moduli spaces of Floer trajectories are canonically isomorphic between respective orbits.*
2. *$K_p < 0$ on $E \subset \widehat{E}$.*
3. *$K_p|_E$ tends to 0 in the C^2 norm on E as p tends to infinity.*

This theorem implies that:

$$\varinjlim_p SH_*^{[0,\infty)}(K_p) = \varinjlim_p SH_*(\varrho_p) \tag{1}$$

$SH_*^{[0,\infty)}(K_p) := SH_*(K_p)/SH_*^{(-\infty,0)}(K_p)$ where $SH_*^{(-\infty,0)}$ is the symplectic homology group generated by orbits of negative action. We also have:

$$\varinjlim_p SH_*(K_p) = \varinjlim_p SH_*^{[0,\infty)}(K_p) \tag{2}$$

This is because there exists a cofinal family of Lefschetz admissible Hamiltonians G_p such that:

(1) $G_p < 0$ on $E \subset \widehat{E}$.
(2) $G_p|_E$ tends to 0 in the C^2 norm on E as p tends to infinity.
(3) All the periodic orbits of G_p have positive action.

Property (3) of G_p will follow from Lemma 5.6. Using the fact that both K_p and G_p are cofinal, tending to 0 in the C^2 norm on E and are non-positive on E, there exist sequences p_i and q_i such that:

$$K_{p_i} \leq G_{q_i} \leq K_{p_{i+1}}$$

for all i. Hence:

$$\varinjlim_p SH_*^{[0,\infty)}(G_p) = \varinjlim_p SH_*^{[0,\infty)}(K_p).$$

Property (3) of G_p implies:

$$\varinjlim_p SH_*^{[0,\infty)}(G_p) = \varinjlim_p SH_*(G_p).$$

This gives us equation (2). Combining this with equation (1) gives:

$$\varinjlim_p SH_*(K_p) = \varinjlim_p SH_*(\varrho_p).$$

This proves Theorem 2.22.

Before we prove Theorem 5.5, we need two preliminary Lemmas. We need a preliminary Lemma telling us something about the flow of a Lefschetz admissible Hamiltonian. We let $H = \pi^* H_S + \pi_1^* H_F$ be as in Definition 2.21. We assume that the slope of H_S and H_F is strictly less than some constant $B > 0$. We set H_F to be zero in F, and H_F to be equal to $h_F(r_F)$ in the region $r_F \geq 1$ such that $h_F'(r_F) \geq 0$ and $h_F''(r_F) \geq 0$. We also assume that for some very small $\epsilon > 0$, h_F' is constant for $r_F > \epsilon$ and not in the period spectrum of ∂F so that all the orbits lie in the region $r_F \leq \epsilon$. We define H_S in exactly the same way so that it is zero in S and equal to $h_S(r_S)$ on the cylindrical end of $\widehat{S}$ where h_S has the same properties as h_F. The action of an orbit of H_F in the cylinder $r_F \geq 1$ is $r_F h_F'(r_F) - h_F(r_F)$ and similarly the action of an orbit of H_S in $r_S \geq 1$ is $r_S h_S'(r_S) - h_S(r_S)$, so we can choose ϵ small enough so that the actions of the orbits lie in the interval $[0, B]$ because the slope of H_S and H_F is less than B. We have from

Section 4. $\theta = \Theta + k\pi^*\theta_S$ where Θ is the 1-form associated to the Lefschetz fibration (it is a 1-form such that $\Theta|_F$ makes each fibre F into a compact convex symplectic manifold. Also θ_S is the 1-form making the base S into a compact convex symplectic manifold. The constant k is some large constant.

Lemma 5.6. *For k large enough, there exists a constant Ξ depending only on E and θ (not on H) such that the action of any orbit of H is contained in the interval $[0, \Xi B]$.*

Proof. Inside E, we have that the Hamiltonian is 0 so all the orbits have action 0 there. In the region A as defined in Definition 2.21, we have that the orbits come in pairs (γ, Γ) where γ is an orbit from H_S and Γ is an orbit of positive action from H_F. The action of (γ, Γ) is the sum of the actions of γ and Γ. Both these actions are positive. Also their actions are bounded above by B.

So we only need to consider orbits outside the region $A \cup E$. The Hamiltonian $\pi_1^* H_F$ is zero in this region so we only need to consider $\pi^* H_S$. We will consider the orbits of $\pi^* H_S$ in the region $r_S \geq 1$. In this region, there are no singular fibres of the Lefschetz fibration, so we have a well defined plane field P which is the ω-orthogonal plane field to the vertical plane field which is the plane field tangent to the fibres of π. The Hamiltonian flow only depends on $\omega|_P$ and not the vertical plane field because $\pi^* H_S$ restricts to zero on the vertical plane field. The symplecic form $\omega|_P$ is equal to $Gk\pi^* d\theta_S|_P$ for some function $G > 0$. This means that the Hamiltonian vector field associated to $\pi^* H_S$ is $\frac{1}{G}$ times the horizontal lift of the Hamiltonian vector field associated to H_S in S. Let V be this horizontal lift. The construction of the completion of a Lefschetz fibration before Definition 2.16 ensures that the region $r_S \gg 1$ is a product $W \times [1, \infty)$ where r_S parameterizes the second factor of this product and Θ is a pullback of a 1-form on W via the natural projection $W \times [1, \infty) \to W$. This means that Θ is invariant under translations in the r_S direction (i.e. under the flow of the vector field $\frac{\partial}{\partial r_S}$ which is $\frac{1}{r_S}$ times the horizontal lift of λ_S where λ_S is the Liouville flow in $\widehat{S}$). We also have that $d\theta_S$ is invariant under translations in the r_S direction (i.e. under the flow of $\frac{1}{r_S}\lambda_S$). Hence the symplectic structure ω is also invariant under translations in the r_S direction for $r_S \gg 0$. This means that the function G is bounded above and below by positive constants as the symplectic structure is invariant under translations in the r_S direction and if we travel to infinity in the fibrewise direction (i.e. if we travel into the region A), then $G = 1$. We want bounds on the function $V(\theta)$ because the function G is bounded. Let Y be the Hamiltonian flow of r_S in $\widehat{S}$ and let $\tilde{Y}$ be its horizontal lift to P. We have that $Y(\theta_S) = 1$. This means that $\tilde{Y}(\pi^*\theta_S) = 1$. We also have that $\tilde{Y}(\Theta)$ is bounded because Θ is invariant in the r_S direction for r_S large and $\tilde{Y}(\Theta) = 0$ if we are near infinity in the fibrewise direction. We choose the constant k large enough so that $\tilde{Y}(\theta) = \tilde{Y}(\Theta) + k\tilde{Y}(\pi^*\theta_S) > 0$. This function is also bounded above because $\tilde{Y}(\Theta)$ is bounded and $k\tilde{Y}(\pi^*\theta_S) = kY(\theta_S) = k$. This choice of k only depends on the Lefschetz fibration and not on H. Now, $V = h'_S(r_S)\tilde{Y}$. Because h'_S bounded below by 0, we have that $V(\theta)$ is bounded below by 0 and bounded above by some constant multiplied by the slope of H_S. All the orbits of H lie in some compact set where H is C^0 small, so the action of an orbit is near $\int_o V(\theta)dx$ where the integral is taken over an orbit o and dx is the volume form on o giving it a volume of 1. This means that the action of these orbits is in the interval $[0, \Xi B]$ for some constant Ξ. This completes our theorem. □

The manifold $\widehat{M} = \widehat{E}$ has a cylindrical end $\partial M \times [1, \infty)$. We let r be the radial coordinate of this cylindrical end. The we define set $\{r \leq R\}$ to be equal to $M \cup (\partial M \times [1, R])$. We define the sets $\{r_F \leq R\}$ and $\{r_S \leq R\}$ in a similar way.

Lemma 5.7. *There exists a constant $\varpi > 0$ such that for all $R \geq 1$, we have that $\{r \leq R\} \subset \{r_S \leq \varpi R\}$ and $\{r \leq R\} \subset \{r_F \leq \varpi R\}$.*

Proof. We will deal with r_S first. The level set $r = R$ is equal to the flow of ∂M along the Liouville vector field λ for a time $\log(R)$. Hence, all we need to do is show that $dr_S(\lambda)$ is bounded above by $e^{\varpi} r_S$. This means that if p is a point in ∂M, then the rate at which $r_S(p)$ increases as we flow p along λ is bounded above by $e^{\varpi} r_S(p)$. Hence if we flow p for a time $\log(R)$ to a point q, then $r_S(q) \leq \varpi R$ which is our result.

We will now show $dr_S(\lambda)$ is bounded above by $e^{\varpi} r_S$ to finish the first part of our proof. We let Θ be a 1-form associated to E as constructed before Definition 2.16. Then $\theta = \Theta + \pi^*\theta_S$ where θ_S is a convex symplectic structure for the base $\widehat{S}$. We have that $\omega = d\Theta + \pi^* d\theta_S$. The construction before Definition 2.16 ensures that the region $r_S \gg 1$ is a product $W \times [1, \infty)$ where r_S parameterizes the second factor of this product and Θ is a pullback of a 1-form on W via the natural projection $W \times [1, \infty) \to W$. This means that Θ is invariant under translations in the r_S direction. Hence $d\Theta$ is also invariant under these translations. Also $\pi^* d\theta_S$ is invariant under translations in the r_S direction. All of this means that the vector field V defined as the ω-dual of Θ is invariant under these translations for r_S large. This implies that $dr_S(V)$ is bounded.

Let V' be the ω-dual of $\pi^*\theta_S$. Let λ_S be the Liouville vector field in $\widehat{S}$. Then $V' = GL$ where L is the horizontal lift of λ_S and $G : \widehat{E} \to \mathbb{R}$ is defined in the proof of Lemma 5.6. The proof of Lemma 5.6, tells us that G is a bounded function. Also, $dr_S(\lambda_S) = r_S$, hence

$$dr_S(V') = G dr_S(L) = G dr_S(\lambda_S) = G r_S$$

Hence $dr_S(V')$ is bounded above by some constant multiplied by r_S. Finally, we have that $\lambda = V + V'$ which means that there exists a $\varpi > 0$ such that $dr_S(\lambda)$ is bounded above by $e^{\varpi} r_S$.

We will now deal with r_F. This is slightly more straightforward because the Lefschetz fibration is a product $\partial F \times [1, \infty) \times \widehat{S}$ and θ splits up in this product as $\Theta + \pi^* \theta_S$, where we can view Θ as 1-form on $\partial F \times [1, \infty)$. We need to bound $dr_F(\lambda)$. In this case, because everything splits in this product, we have that $dr_F(\lambda) = dr_F(\Lambda)$ where Λ is the ω-dual of Θ. This is equal to $r_F \leq e^{\varpi} r_F$ as $\varpi > 0$. Hence we have that $r \leq R$ implies that $r_F \leq \varpi R$. □

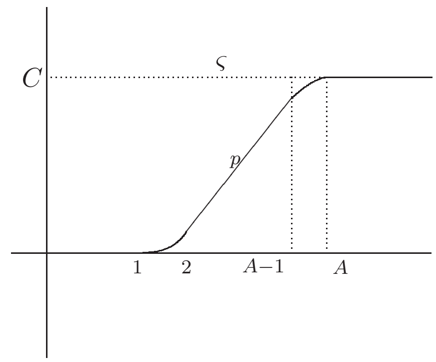
Proof. of Theorem 5.5. Let ϱ_p be the Hamiltonian as above. We will write $\varrho = \varrho_p$ for simplicity. The idea of the proof is to modify the Hamiltonian ϱ outside some large compact set so that it becomes Lefschetz admissible and in the process only create orbits of negative action without changing the orbits of ϱ or the Floer trajectories connecting orbits of ϱ. We will do this in three sections. In section (a), we will modify ϱ to a Hamiltonian ς so that it becomes constant outside a large compact set κ while only adding orbits of negative action. This is exactly the same as the construction due to Hermann [15]. In section (b) we will consider a Lefschetz admissible Hamiltonian L which is 0 in the region κ, but has action bounded above so that the orbits of $L + \varsigma$ outside κ have negative action. We define our cofinal family $K_p := L + \varsigma$. (c) we ensure that the Floer trajectories and pairs of pants satisfying Floer's equation connecting orbits of positive action stay inside the region $r \leq 2$.

(a) We have that p is the slope of the Hamiltonian ϱ and this is not in the period spectrum of ∂M. Hence, we define $\mu := \mu(p) > 0$ to be smaller than the distance between p and the action spectrum. Define:

$$A = A(p) := 3p/\mu > 1.$$

We can assume that $A > 4$ because we can choose μ to be arbitrarily small. Remember that $\widehat{E} = \widehat{M}$ where M is a compact convex symplectic manifold, and that r is the radial coordinate for the cylindrical end of $\widehat{M}$. We define ς to be equal to ϱ on $r \leq A - 1$. On the region $r \geq 1$, we have that ϱ is equal to $h_p(r)$. We will just write h instead of h_p. Set $\varsigma = k(r)$ for $r \geq 1$ with non negative derivative. This means that in the region $1 \leq r \leq A - 1$ we have that $h(r) = k(r)$. Hence in $r \leq A - 1$ we have that $k''(r) \geq 0$ and $k'(r) \geq 0$, and in the region $2 \leq r \leq A - 1$ we have $k'(r) = p$. Also we have that ς is C^2 small and negative for r near 1. Because ς is C^2 small, we can also assume that p is large enough so that for r near 1, $k' \ll p$. Because ϱ_p is cofinal, we can assume that p is large enough so that $h(2) = k(2) > 0$. Both these previous facts mean that $p(A-2) < k(A-1) < p(A-1)$. Outside this region, we define k to be a function with the following constraints: For $r \geq A$ set $k(r)$ to be constant and equal to C where $C = p(A-1)$. In the region $A-1 \leq r$, $k'' \leq 0$. We assume that $k' \geq 0$ for all $r \geq 1$. Here is a picture:

Figure 5.8.



We want to show that the additional orbits of ς only have negative action. All these orbits lie in the region $r \geq 2$. In fact because p is not in the action spectrum, they lie in the region $r \geq A-1$. In the region $\{r : p-\mu < k'(r) \leq p\}$, we have that ς has no periodic orbits. Also, the action of a periodic orbit is $k'(r)r - k(r)$. Combining these two facts implies that the action of a periodic orbit in the region $2 \leq r$ is less than

$$(p-\mu)r - k(r) \leq (p-\mu)A - p(A-2)$$

$$= -\mu A + 2p = -\mu \frac{3p}{\mu} + 2p = -p < 0$$

Hence we have a Hamiltonian ς equal to ϱ in the region $r \leq 2$ and such that it is constant and equal to $C = p(A-1)$ in the region $r \geq A-1$ and such that all the additional periodic orbits created have negative action.

(b) Lemma 5.6 tells us that there exists a cofinal family of Lefschetz admissible Hamiltonians Λ_p such that the action spectrum of Λ_p is bounded above by some constant Ξ multiplied by the slope of λ_p. We can assume that both the slopes of λ_p are equal to $\sqrt(p)$ (if $\sqrt{p}$ is in the action spectrum of the fibre or the base, then we perturb this value slightly to ensure that Λ_p has orbits in a compact set). This means that the action of Λ_p is bounded above by $\Xi\sqrt{p}$. The Hamiltonian Λ_p is equal to zero in E. We will now define a Hamiltonian L_p as follows: We let ϖ be defined as in Lemma 5.7. Set $L_p = 0$ in the region $\{r_S \leq \varpi A\} \cap \{r_F \leq \varpi A\}$. In the region $\{r_S \geq 1\} \cup \{r_F \geq 1\}$, we have that Λ_p is a function of the form $\pi_1^* h_F(r_F) + \pi^* h_S(r_S)$. Here, π_1 is the natural projection: $\partial F \times [1,\infty) \times \widehat{S} \to \partial F \times [1,\infty)$ (this is the same as the projection defined just before Definition 2.21). So, we set the function $\pi_1^* h_F(r_F)$ to be zero outside the domain of definition of π_1. Also, $\pi^* h_S$ is zero outside the region $r_S \geq 1$. We define L_p to be

$$\pi_1^* h_F(r_F - \varpi A) + \pi^* h_S(r_S - \varpi A)$$

in the region $\{r_S \geq \varpi A\} \cup \{r_F \geq \varpi A\}$. Hence we have a well defined function L_p. Because L_p has scaled up, we have that the action spectrum of L_p is equal to ϖA multiplied by the action spectrum of Λ_p. Hence, we have that the action spectrum of L_p is bounded above by $\varpi A \Xi \sqrt{p}$.

Because $\{r \leq A\} \subset \{r_S \leq A\} \cap \{r_F \leq A\}$, we can add L_p to ς without changing the orbits of ς in the region $r \leq A$. Also, the action of the orbits of $\varsigma + L_p$ in the region $r \geq A$ is bounded above by $\varpi A \Xi \sqrt{p} - p(A-1)$. So for p large enough we have that the additional orbits added are of negative action.

(c) We choose an almost complex structure $J \in \mathcal{J}^h(\widehat{E})$ such that on some neighbourhood of the hypersurface $r = 2$, J is admissible. Then [3, Lemma 7.2] and the comment after this Lemma ensure that no Floer trajectory or pair of pants satisfying Floer's equation connecting orbits inside $r < 2$ can escape $r \leq 2$. Hence our Hamiltonian $K_p := \varsigma + L_p$ has all the required properties.

5.1. **A better cofinal family for the Lefschetz fibration.** In this section we will prove Theorem 2.24. We consider a compact convex Lefschetz fibration (E, π) fibred over the disc $\mathbb{D}$. Basically the cofinal family is such that $H_F = 0$. This means that the boundary of F does not contribute to symplectic homology of the Lefschetz fibration. The key idea is that near the boundary of F the Lefschetz fibration looks like a product $\mathbb{D} \times \text{nhd}(\partial F)$ and because symplectic homology of the disc is 0 we should get that the boundary contributes nothing. Statement of Theorem 2.24:

$$SH_*(E) \cong SH_*^{\text{lef}}(E).$$

We will define $F, S(= \mathbb{D}), r_S, r_F, \pi_1$ as in the previous section. This means that the compact convex sympectic manifold F is a fibre of E and S is the base which in this section is equal to $\mathbb{D}$. We also have that r_S is a radial coordinate for the cylindrical end of $\widehat{S}$ which we also identify with $\pi^* r_S$. The map π_1 is the natural projection $(\partial F \times [1, \infty)) \times \widehat{S} \twoheadrightarrow (\partial F \times [1, \infty))$ where $(\partial F \times [1, \infty)) \times \widehat{F}$ is a subset of $\widehat{E}$. The function r_F is a radial coordinate for the cylindrical end of $\widehat{F}$ which we also identify with $\pi_1^* r_F$. Before we prove Theorem 2.24, we will write a short lemma on the $\mathbb{Z}$ grading of $SH_*(E)$.

Lemma 5.9. *Let $\widehat{F} := \pi^{-1}(a) \subset \widehat{E}$ ($a \in \mathbb{D}$). Suppose we have trivialisations of $\mathcal{K}_{\widehat{E}}$ and $\mathcal{K}_{\widehat{S}}$ (these are the canonical bundles for $\widehat{E}$ and $\widehat{S}$ respectively); these naturally induce a trivialisation of $\mathcal{K}_{\widehat{F}}$ away from F. If we smoothly move a, then this smoothly changes the trivialisation.*

Proof. of Lemma 5.9.

We choose a $J \in \mathcal{J}^h(E)$. The bundle E away from E^{crit} has a connection induced by the symplectic structure. Let $A \subset \widehat{E}$ be defined as in Definition 2.21. Let U be a subset of A where

(1) π is J holomorphic.