18.100C Recitation Homework 3

Due on Stellar by 11:59 PM on **Saturday**, October 2

As for most weeks, you will have to turn in one of your homework problems that is due for class to your recitation lecturer through Stellar, by 5 pm on the day that it is due in class.

The Saturday assignment for this week asks that you choose from one of the following topics and write a 2–3 page paper about it. The paper should present the material in a readable manner, keeping in mind our discussion of *motivation* and *guiding text*. The intended audience is a technical one, e.g. a fellow mathematician who happens to pick up your paper.

Your paper will be peer reviewed, meaning a fellow classmate will read and comment on your writing. Because of this, we are trying to get an even spread of topics. To this end we ask that you submit by **Sunday, September 26** your preferred topic and your second choice topic; submit your preferences by e-mail to your recitation instructor. Topics will be given out on a first-come, first-served basis.

1. **Cardinality of connected spaces:** (Exercise 2.19 in Rudin) Let X be a connected metric space with at least two points. Prove that X is uncountable. Here is a suggested outline.
   (a) If A and B are disjoint closed sets in X, prove that they are separated.
   (b) Prove that the result of part **(a)** holds also for disjoint open sets A and B.
   (c) Fix $x \in X$ and $\delta > 0$. Let $B = B_\delta(x)$, and let $A \subseteq B^c$ be the set $A = \{y \in X\,;\, d(x,y) > \delta\}$. Prove that A and B are separated.
   (d) Use the uncountability of the positive real numbers to conclude that X is uncountable.

2. **Spherical neighbourhood:** Consider the metric space $\mathbb{R}^n$ equipped with its usual Euclidean metric. For any point $x \in \mathbb{R}^n$, and any $r > 0$, let $S_r(x)$ denote the *sphere* of radius r centred at x:
$$S_r(x) = \{y \in \mathbb{R}^n\,;\, d(x,y) = r\}.$$
   Let $X \subseteq \mathbb{R}^n$ be any subset, and define $S_r[X]$ to be the union of all r-spheres centred at points of X:
$$S_r[X] = \bigcup_{x\in X} S_r(x).$$

Show that $S_r[X]$ is compact if X is compact. [*Hint*: it may be useful to consider the alternate definitions of compactness suggested by Rudin's Theorem 2.41 (Heine-Borel) rather than the original Definition 2.32.]

3. **Separable spaces:** A metric space is separable if it contains a countable or finite dense subset (this differs slightly from Rudin's definition in Exercise 2.22). Let X be a separable metric space, and let Y be a subset of X. Prove that Y is a separable metric space. [Hint: let D be a countable (or finite) dense subset of X, and consider all balls with radii $1/n$ centred at points of D, for n ranging through the natural numbers.]