ON AN EFFICIENT SQUARE-ROOT ALGORITHM

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In this paper, we prove the correctness of an algorithm for finding square roots.

Theorem 1. *For any positive number α and any $x_1 > \sqrt{\alpha}$, we may define a sequence $\{x_2, x_3, \ldots\}$ by the recursion formula*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

Then that $\{x_n\}$ decreases monotonically and $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

Proof. We prove this by two inductions.

Claim 1: $x_n > \sqrt{\alpha}$ for all n. To see this, note it holds for x_1, and if $x_k > \sqrt{\alpha}$, then

$$\begin{aligned} x_{k+1} - \sqrt{\alpha} &= \frac{1}{2}\left(x_k + \frac{\alpha}{x_k}\right) - \sqrt{\alpha} \\ &= \frac{1}{2}(x_k - \sqrt{\alpha})\left(1 - \frac{\sqrt{\alpha}}{x_k}\right) \\ &> 0. \end{aligned}$$

Claim 2: $x_{n+1} < x_n$ for every n. To see this, first note if $x_1 > \sqrt{\alpha}$, then

$$\begin{aligned} x_2 &= \frac{1}{2}\left(x_1 + \frac{\alpha}{x_1}\right) \\ &< \frac{1}{2}\left(x_1 + \frac{\alpha}{\sqrt{\alpha}}\right) \\ &= \frac{1}{2}\left(x_1 + \sqrt{\alpha}\right) \\ &< x_1. \end{aligned}$$

If $x_k < x_{k-1}$, then

$$\begin{aligned} x_k - x_{k+1} &= \frac{1}{2}\left(x_{k-1} - x_k + \frac{\alpha}{x_{k-1}} - \frac{\alpha}{x_k}\right) \\ &= \frac{1}{2}(x_{k-1} - x_k)\left(1 - \frac{\alpha}{x_k x_{k-1}}\right) \\ &> 0, \end{aligned}$$

since $x_k x_{k-1} > \alpha$ by Claim 1.

The sequence x_n is monotone decreasing and bounded below, therefore converges. To find the limit, note that x_{n+1} and x_n converge to the same thing, say x. Then we have

$$x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right),$$

or $x = \sqrt{\alpha}$. □

Theorem 2. *Define* $\epsilon_n = x_n - \sqrt{\alpha}$. *Then*

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}},$$

and so if we set $\beta = 2\sqrt{\alpha}$ *we have*

$$\epsilon_{n+1} < \beta \left(\frac{\epsilon_1}{\beta}\right)^{2^n}.$$

Proof. It was show in the proof of Theorem 1 that

$$\begin{aligned}
x_{n+1} - \sqrt{\alpha} &= \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} \\
&= \frac{1}{2}\left(x_n - 2\sqrt{\alpha} + \frac{\alpha}{x_n}\right) \\
&= \frac{1}{2}\left((x_n - \sqrt{\alpha}) + \left(-\sqrt{\alpha} + \frac{\alpha}{x_n}\right)\right) \\
&= \frac{1}{2}\left(x_n\left(1 - \frac{\sqrt{\alpha}}{x_n}\right) - \sqrt{\alpha}\left(1 - \frac{\sqrt{\alpha}}{x_n}\right)\right) \\
&= \frac{1}{2}(x_n - \sqrt{\alpha})\left(1 - \frac{\sqrt{\alpha}}{x_n}\right) \\
&= \frac{1}{2}\frac{(x_n - \sqrt{\alpha})^2}{x_n} \\
&= \frac{\epsilon_n^2}{2x_n},
\end{aligned}$$

and $x_n > \sqrt{\alpha}$ for every n, which gives the stated inequality. The rest follows by induction. □