APPENDIX

The Use of Symbols: A Case Study

In this appendix, I take a symbol-laden article and show how it can be drastically simplified. Each expression to be replaced is enclosed in large parentheses and, for comparison, each suggested replacement follows immediately and is enclosed in square brackets. As a result, one can appreciate the simplification even without keeping track of the mathematical details. If you look at nothing else, at least take a moment to compare equations (11) and (12) with their counterparts (11′) and (12′).

The mathematics itself, while sophisticated, is elementary. There is no algebra, no geometry, no convergence; essentially, there is only bookkeeping.

Recall the three important steps to take toward simplifying your notation:

I. Use an uncomplicated symbol in place of an elaborate one.
II. Discard any symbol that is just plain unnecessary.
III. Simplify the mathematical argument itself.

The article I picked is Sierpinski [24], as corrected by K. Kunugui. It is the source of the monster symbol displayed in Section 5.6, and contains examples of all three types of problems.

In order to eliminate distractions, I have streamlined a great deal of the notation. In particular, I have recast the entire article in terms of **R**, the real numbers, so that the setting will be familiar to everyone and the argument elementary. (There are some comments about this at the end of the discussion.) The theorem now reads as follows:

THEOREM. Assume there exists a family $(A_\alpha)_{\alpha\in\mathbf{R}}$ of countably infinite subsets of **R** such that, for $\alpha \neq \beta$, either $\alpha \in A_\beta$ or $\beta \in A_\alpha$. Then there exists a sequence of functions $f_k\colon \mathbf{R} \to \mathbf{R}$ such that every uncountable set is mapped onto **R** by all but finitely many f_k.

REMARK. Note that the assumed family is indexed by **R** itself.

Proof. The only special fact needed in the proof is that **R** can be put into one-one correspondence with the set of all pairs of sequences $n_1, n_2, \ldots, t_1, t_2, \ldots$, in which (n_k) is an increasing sequence in **N** (the natural numbers) and (t_k) is a sequence in **R**. Consider any $\alpha \in \mathbf{R}$. To indicate that α corresponds to the pair (n_k), (t_k), Sierpinski writes

$$n_k = n_k^\alpha \quad \text{and} \quad t_k = t_k^\alpha; \tag{1}$$

and he enumerates the countably infinite set A_α as

$$\Big(\quad A_\alpha = \{\xi_1^\alpha, \xi_2^\alpha, \ldots\}. \quad \Big) \tag{2}$$

$$\Big[\quad A_\alpha = \{\alpha_1, \alpha_2, \ldots\}. \quad \Big] \tag{2$'$}$$

(The alternative (2′) is rid of the extra letter ξ.) (**Type I.**) Then he defines $f_k(\alpha)$ by the monster symbol. Sierpinski had a pixyish quality and may have done all this partly because it was so much fun. But as it turns out, Sierpinski's argument is not complicated *enough*, and at the end of his paper, at the decisive step, he confuses a sequence with a subsequence.

Kunugui corrects the proof as follows. First, inductively, he picks an integer

$$\Big(\quad l_i^\alpha \in \{n_1^{\xi_i^\alpha}, n_2^{\xi_i^\alpha}, \ldots\} \quad (i = 1, 2, \ldots) \quad \Big) \tag{3}$$

$$\Big[\quad l_i^\alpha \in \{n_1^{\alpha_i}, n_2^{\alpha_i}, \ldots\} \quad (i = 1, 2, \ldots) \quad \Big] \tag{3$'$}$$

so that

$$l_i^\alpha > l_j^\alpha$$

for all $j < i$. This is possible because the sequence in braces is increasing. (Note that (3′) results from (3) by means of (2′).) Then he points out that

$$\Big(\quad l_i^\alpha = n_{j_i^\alpha}^{\xi_i^\alpha}, \quad \Big) \tag{4}$$

$$\Big[\qquad\qquad \Big]$$

where

$$\Big(\quad (j_i^\alpha) \text{ is a sequence of positive integers.} \quad \Big)$$

$$\Big[\qquad\qquad\qquad\qquad \Big]$$

The blank spaces signify that the author's expressions should be replaced by nothing at all—in other words, that what the author is writing is unnecessary. For, as it turns out, he is going to fix α, then determine i, and then, for the one and only time, pick

$$j_i^\alpha.$$

In these circumstances, we can just call it j. (**Type II.**)

Next, Kunugui defines a new sequence of real numbers,

$$\eta_1^\alpha, \eta_2^\alpha, \ldots,$$

as follows:

$$\left(\quad \eta_1^\alpha = \cdots = \eta^\alpha_{l_1^\alpha} = \xi_1^\alpha, \quad \right)$$

$$\left[\qquad \right]$$

and, for $i > 1$,

$$\left(\quad \eta^\alpha_{l_{i-1}^\alpha + 1} = \cdots = \eta^\alpha_{l_i^\alpha} = \xi_i^\alpha . \quad \right) \tag{5}$$

$$\left[\qquad \right]$$

The reason none of this is necessary is that the only numbers

$$\eta_k^\alpha$$

that he uses in the proof are for those k of the form

$$k = l_i^\alpha$$

—but in that case we have, simply,

$$\eta_k^\alpha = \xi_i^\alpha \; [= \alpha_i], \tag{6}$$

without further ado. **(Type III.)**

Finally, the author defines

$$\left(\quad f_k(\alpha) = t^{\eta_k^\alpha}_{n_k^{\eta_k^\alpha}} \qquad (k = 1, 2, \ldots). \quad \right) \tag{7}$$

$$\left[\quad f_k(\alpha) = t_k^{\alpha_i} \qquad (k = l_i^\alpha; \quad i = 1, 2, \ldots). \quad \right] \tag{7'}$$

In (7′), the remaining values of $f_k(\alpha)$ are arbitrary—for definiteness, say

$$f_k(\alpha) = \alpha \qquad (k \neq l_i^\alpha).$$

This change from (7) to (7′), the most significant of all, proceeds in two steps. The first step is simply the substitution (6), which reduces (7) to

$$f_k(\alpha) = t^{\alpha_i}_{n_k^{\alpha_i}} .$$

From here, a certain amount of experimenting leads to (7′). That's the interesting thing about this kind of editing: once you have simplified the mathematics and the notation so that the argument is easier to follow, you may see a way of simplifying the reasoning still further.

Let us now follow through the argument that the functions f_k, thus defined, satisfy the requirements of the theorem. Contrapositively, we consider any set S that infinitely many f_k fail to map onto $\mathbf{R}$, and show that it must be countable. (The original article argues by "questionable" contradiction; see Section 3.8.) By assumption, then, there exist an increasing sequence of indices $n_1, n_2, \ldots$ and a sequence $t_1, t_2, \ldots$ in $\mathbf{R}$ such that

$$t_{n_j} \notin f_{n_j}(S) \qquad (j = 1, 2, \ldots). \tag{8}$$

(The values of t_i for $i \neq n_j$ do not enter into the argument). By (1), there exists $\beta \in \mathbf{R}$ such that

$$\left(\quad n_k = n_k^{\beta}, \qquad t_k = t^{\beta}_{n_k} \quad \right) \tag{9}$$

$$\left[\quad n_k = n_k^{\beta}, \qquad t_k = t_k^{\beta} \quad \right] \tag{9′}$$

for $k = 1, 2, \ldots$. (The little twist in the second half of (9) requires some pondering. The more natural (9′) is what leads to the simplification (7′).)

Now consider any $\alpha \notin A_\beta$, with $\alpha \neq \beta$. (*This is where we fix* α.) By hypothesis, $\beta \in A_\alpha$. Hence (by (2)) [by (2′)], there is an index i for which

$$\left(\quad \beta = \xi_i^{\alpha} . \quad \right)$$

$$\left[\quad \beta = \alpha_i . \quad \right]$$

(*This is where we determine* i.) Consider

$$k = l_i^{\alpha} .$$

In view of (5),

$$\left(\quad \beta = \eta_k^{\alpha} . \quad \right) \tag{10}$$

$$\left[\qquad \right]$$

(Using (4) and (9), we get) [Using (3′) and (9′), we get, for suitable j,]

$$\left(\quad k = l_i^{\alpha} = n^{\xi_i^{\alpha}}_{j_i^{\alpha}} = n^{\beta}_{j_i^{\alpha}} = n_{j_i^{\alpha}} . \quad \right) \tag{11}$$

$$\left[\quad k = l_i^{\alpha} = n_j^{\alpha_i} = n_j^{\beta} = n_j . \quad \right] \tag{11′}$$

(*This is the one time we pick* j.)

Next, from ((9), (10), and (7)) [(9′) and (7′)], we have

$$\left(\quad t_k = t^{\beta}_{n_k} = t^{\beta}_{n_k^{\beta}} = t^{\eta_k^{\alpha}}_{n_k^{\eta_k^{\alpha}}} = f_k(\alpha). \quad \right) \tag{12}$$

$$\left[\quad t_k = t_k^{\beta} = t_k^{\alpha_i} = f_k(\alpha). \quad \right] \tag{12′}$$

Substituting from ((11) into (12)) [(11′) into (12′)] gives us

$$\left(\quad t_{n_{j_i^{\alpha}}} = f_{n_{j_i^{\alpha}}}(\alpha). \quad \right) \tag{13}$$

$$\left[\quad t_{n_j} = f_{n_j}(\alpha). \quad \right] \tag{13′}$$

In view of (8), then, $\alpha \notin S$.

We have shown that for $\alpha \neq \beta$, if $\alpha \notin A_\beta$ then $\alpha \notin S$; that is, if $\alpha \in S$ then $\alpha \in A_\beta$. Since A_β is countable, S is countable. ♦

Comment on the theorem. The original article uses the language of transfinite ordinals; the hypothesis of the theorem is the *continuum hypothesis* (CH), which states that the uncountable sets in **R**, and the set of all countable ordinals, all have the same cardinal as **R** itself. The hypothesis as stated in this Appendix is an adaptation of (CH) to the particular problem. If we assume (CH), then the ordered set S consisting of the negative integers followed by the countable ordinals can be indexed by **R**, and the sets $A_\alpha = \{x \in S: x < \alpha\}$ are as stated. The converse can also be proved directly, but note that it is immediate from the conclusion of the theorem: every uncountable set in **R** is carried by some f_k onto **R**, hence must have the cardinal of **R**.

English dictionaries

[18] *The American Heritage Dictionary of the English Language*, First Edition, Unabridged, 1550 pp. Houghton Mifflin, 1969.

In a class by itself; reputed to have been written as the answer to Webster's permissive Third International. The usage notes are particularly valuable. But I understand that later editions have been watered down.

[19] *The Random House Dictionary of the English Language*, 2059 pp. Random House, 1983.

More comprehensive than *Heritage*. Includes helpful dictionaries to and from French, Spanish, Italian, and German. Exactly twenty years ago this month, I wrote the publisher that the interval [heimisch, Heimsuchung] should *precede* [heimtückisch, heissen] (taking the occasion to allude to the title of the work); the mislocation has withstood all intervening printings.

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