Are there gaps in the rationals? Version %

There is a sense in which there is no empty space in the rational numbers: no matter how close together two rationals are, there are infinitely many other rationals between them. In this sense, there are no gaps in the rationals.

However, there is a sense in which there *is* empty space in the rational numbers. Let

$$A = \left\{ r \in \mathbb{Q} : r > 0 \text{ and } r^2 \leq 2 \right\} \quad \text{and} \quad B = \left\{ s \in \mathbb{Q} : s > 0 \text{ and } s^2 \geq 2 \right\}.$$

Clearly, any element of A is less than any element of B. Nonetheless, there is no element $q \in \mathbb{Q}$ such that q is between A and B. Thus it is not always possible to find a rational number between any two *sets* of rationals. In this sense, there are gaps in the rationals.

To make matters geometrically explicit, consider the graph of $y = 2 - x^2$ using only rational numbers. Notice that there's no $x \in \mathbb{Q}$ that makes $2 - x^2 = 0$. Thus, the graph never crosses the x-axis, despite the fact that the graph is above the x-axis at $x = 0$ and below the x-axis at $x = 3$. This demonstrates the existence of a more devious type of gap in the rationals, and the failure of $\mathbb{Q}$ to model our intuition about the physical world.

Show that every infinite subset of $[0,1] \times [0,1]$ *has a limit point.*

Let $K = [0,1] \times [0,1]$ be the unit square and let $S \subseteq K$. We proceed by proving the contrapositive: If S has no limit points, then S is finite.

Suppose that S has no limit points. Then for each $p \in K$, there exists some ball around p not containing any point of S, except perhaps possibly for p itself. Finitely many of these balls cover K, and this cover also covers S. Thus, S is finite because it can be covered by finitely many balls. Taking the contrapositive, if S is an infinite subset of $[0,1] \times [0,1]$, then S has a limit point.

Are there gaps in the rationals? Version

There is a sense in which there is no empty space in the rational numbers: no matter how close together two rationals are, there are infinitely many other rationals between them. In this sense, there are no gaps in the rationals.

However, there is a sense in which there *is* empty space in the rational numbers: while it is possible to find a rational number between any two other rational numbers, it is *not* always possible to find a rational number between any two *sets* of rationals. For example, let

$$A = \left\{r \in \mathbb{Q} : r > 0 \text{ and } r^2 \leq 2\right\} \quad \text{and} \quad B = \left\{s \in \mathbb{Q} : s > 0 \text{ and } s^2 \geq 2\right\}.$$

Clearly, any element of A is less than any element of B. Nonetheless, there is no element $q \in \mathbb{Q}$ such that q is between A and B. In this sense, there is a gap in the rationals between sets A and B.

To make matters geometrically explicit, consider the graph of $y = 2 - x^2$ using only rational numbers. At $x = 0$, the graph is above the x-axis, and at $x = 3$ it is below the x-axis. However, the graph never actually crosses the x-axis, because there's no $x \in \mathbb{Q}$ that makes $2 - x^2 = 0$. This demonstrates the existence of a more devious type of gap in the rationals, and the failure of $\mathbb{Q}$ to model our intuition about the physical world.